NEW S-FUNCTION SERIES AND APPLICATION OF GROUP THEORY IN SUPERSYMMETRIES

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To My Parents

CONTENTS

LIST OF TABLES AND FIGURES

ABSTRACT

This thesis is devoted to the study of S-function series and the application of group theory in two physical problems: the decomposition of the basic spin irreps of the extended Poincaré supersymmetry and the construction of a colour superalgebra containing generalized quasispin subalgebra, as a receptacle for the dynamical algebra U(M/N) of nuclei supersymmetry.

The sixteen classical S-function series which have long been recognized as important in obtaining the branching rules and Kronecker product rules for the classical Lie groups, are classified into six families of three types. Families of new series of type I and III are generated. Techniques are developed to identify the S-function contents of the new series. More than forty new series with well defined generating functions and standard S-function expansions are listed. Possible applications of the new series are mentioned.

S-function techniques are used to obtain branching rules for the basic spin irreps of the special orthogonal group SO_{2k} under the restriction $SO_{2k} \downarrow SO_{D-2} \times K$ where D is the spacetime dimension of the extended D-dimensional Poincaré supersymmetry and K is the appropriate automorphism group which is D-dependent. General results for $D \leq 10$ capable of extending to decompositions of irreps giving rise to helicities greater than two are given together with a general method for $D > 10$. A number of explicit decompositions are tabulated. Formulas for calculating spin plethysms of SO_n (for $n < 10$) to any order are given. Several new branching rules for subgroups of SO_{2k} are developed.

Dynamical supersymmetries in nuclei and the harmonic oscillator (boson, fermion) realizations of classical Lie algebras and superalgebras

are reviewed. A non-compact $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ graded colour superalgebra SpO(2M/1/2N/0) constructed out of the supercreation-annihilation operators is identified as a receptacle for the dynamical superalgebra U(M/N) of nuclei supersymmetry. Various subalgebras of the big algebra are discussed. The existence of a generalised quasispin algebra is demonstrated and its applications discussed.

INTRODUCTION

Symmetry considerations have played a dominant role in the development of all branches of physics. The importance of group theory and its utility is now well established and universally recognized. It has become part of the ordinary mathematical equipment of physicists.

The applications of group theory can be subdivided into two broad areas, one where the underlying dynamical laws are known, the other where they are as yet unknown and only the kinematical symmetries can serve as a certain guide. In the first case group theoretical techniques are used to exploit the known symmetries, either to simplify numerical calculations or to obtain exact analytic results. In the second area group theory plays a more profound role: it is used to discover as much as possible of the underlying symmetries and, through them, learn about the physical laws of interactions. The study of nuclear structures and elementary particles belong to this second category.

Recently it has been suggested that there may occur in nature more complex types of symmetries. In contrast to ordinary symmetry which applies to systems of pure bosons or pure fermions, the new supersymmetry would apply to mixed systems of bosons and fermions. Supergroups and superalgebras have been used to construct models in various areas of physics including elementary particle theory, condensed matter physics, nuclear structure physics and supergravity.

In part, as a consequence of these developments, physicists have been forced to concern themselves more with mathematical aspects of the theory of groups. A wide class of generalized Lie algebras and groups have been described, the classification and representation theories considered and various techniques have been developed to carry out calculations of group representations.

The S-function techniques (or Young tableaux method) provide a particularly convenient means of performing calculations such as Kronecker products dimension of representation and branching rules, not only for semi-simple Lie groups, but also for classical Lie supergroups that arise in the applications of group theory to physics. In contrast to the more common weight vector approach, the S-function approach has the merit that it gives succinct results in a rank-independent fashion. The most important contributions to this subject were made by Littlewood (1950). Subsequent developments were due to various mathematical physicists (Wybourne 1970, King 1975, King, Luan and Wybourne 1981, Black, King and Wybourne 1983). S-function series play an important integral part in this development.

This thesis is concerned with three problems, namely, developing new S-function series; obtaining branching rules that arise in the extended Poincaré supersymmetry and finally, constructing a generalized algebra which is a receptacle of the nuclear dynamical superalgebra U(M/N) and contains a generalized quasispin subalgebra. For convenience, the thesis has been divided into three chapters.

The first chapter is devoted to the development of new S-function series. The basic notations such as partitions of integers, the Frobenius notation, Young diagrams of partitions and symmetric functions are introduced. The S-function is then defined as a bialternant and denoted by partitions. This is the most convenient form for discussing S-function series as to be seen later on. Several equivalent definitions of S-functions are also given which serve to show the relationship between S-functions and other symmetric functions and especially to the characters of the unitary group. S-function operations and modification rules are outlined. These are followed by the introduction of S-function series and the definition of inverse, conjugate and adjoint series. The sixteen classical S-function series are summarized and classified, according to the form of their generating functions, into 6 families of three types. The relations betwen generating function for a series of type I, II or III and that for the inverse, conjugate and adjoint series are examined. New S-function series are then generated either by substitutions in L-family series or by taking products of known series. The new series obtained are either of type I or III and can be regarded as outer plethysms of special S-functions with the L-family series. S-function contents of the new series are then established using various techniques, especially the determinant method first used by Littlewood in establishing the A and C series. In order to convert the non-standard S-function contents of the new series systematically into standard forms, the idea of 'standard elementary set', 'joint' and 'hook' of S-functions and 'order of precedence' are introduced. More than forty new S-function series with well defined generating functions and standard S-function contents have been tabulated. With the techniques developed in this chapter other new series can be found if desired.

The second chapter is concerned with a problem that arises in the extended Poincaré supersymmetry, namely, the decomposition of the basic spin irrep Δ or $\Delta_{\pm}$ of SO_{2k} into the subgroup $SO_{D-2} \times K$, where K can be $SU_N \times U_1$, SO_N or Sp_N depending upon the spacetime dimension D, N is to be fixed by the value of k and D. S-function techniques are used for solving the problem. The isomorphisms and automorphism of SO_n for $n \leq 8$ are examined and employed to obtain general formulas for the plelthysms of the basis spin irreps of SO_n for $n \leq 9$. Explicit results for antisymmetric m-th powers of the basic spin irreps of SO_n for $n \leq 11$, $m \leq 8$ and symmetrized powers for $n \leq 10$, $m \leq 4$ are tabulated, a general theorem on S-function plethysm and branching rule is quoted. Reality types of the basic spin irreps are reviewed since they are important in determining the automorphism group K. A number of additional branching rules for important subgroups of SO_{2k} were then established by making use of the known Kronecker products, branching rules and S-function series identities. Following that the branching rules for the basic spin irreps under $SO_{2k} \downarrow SO_{D-2} \times K$ for $D \leq 10$ are obtained by first specifying the decomposition of the vector irrep of SO_{2k} and then using the 'plethysm branching rule theorem' and the local isomorphisms and outer automorphism of SO_{D-2} for $D \leq 10$. For $D > 10$, by considering the equivalent two step decomposition a general method is described which can be used to obtain the desired branching rules. Explicit decompositions for $D \leq 11$ have been tabulated.

Chapter three deals with a problem of different nature. The general idea of boson fermion realization of classical Lie algebras and superalgebras, dynamical symmetry, IBM and IBFM of nuclei are reviewed. After the grading of super operators is defined, a $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ graded colour superalgebra Sp(2M/1/0/2N) is constructed. Various subalgebras of the big algebra including the nuclei dynamical supersymmetry algebra U(M/N) and a generalized quasispin algebra Sp(2,R) have been identified. Properties of the generalized quasispin algebra are examined.

CHAPTER I

NEW S-FUNCTION SERIES

I.1 INTRODUCTION

In the S-function approach to group theory, S-function series have proved to be very important in that they provide succinct expressions for many operations such as Kronecker products and branching rules and are a useful aid for symbolic manipulations of group characters.

The importance of some particular S-function series in the study of classical groups were first discovered by Littlewood (1950). It was shown that these series can be used to link characters of classical Lie groups to S-functions (or characters of the unitary group) on which various operations can be applied. For example, the character of symplectic group can be written as $\langle\lambda\rangle = \{\lambda/A\}$ or conversely the character of unitary group (or S-function) can be expressed as $\{\lambda\} = \langle\lambda/B\rangle$ where A and B are special S-function series.

The aim of this chapter is to derive new S-function series which may be potentially important in applications.

The arrangement of this chapter is as follows:

In section 4.2, basic notations of partitions and S-functions are introduced, S-function operations and modification rules outlined.

In section 4.3, the sixteen classical S-function series are classified into six families of three types. Properties of these series are summarized.

The preceding work sets the scenario for a discussion of new S-function series. In section 4.4, more than forty new S-functions are generated, their S-function content derived using various techniques. The results are displayed in tables.

I.2 BASIC THEORY OF S-FUNCTIONS

The S-function is usually denoted as $\{\lambda\}$ or λ where $(\lambda) = (\lambda_1, \lambda_2, \ldots, \lambda_p)$ is a *partition* of some integer n. Basic definitions of partitions, frames of partitions and Frobenius notation is given in I.2.1.

The S-function is a special kind of symmetric polynomial function. It was first discovered by Jacobi in 1841, and studied by Jacobi, Trudi, Naegelsbach, Kostka, and later on developed by Littlewood and Macdonald. Since it was originally defined as ratio of two alternants, the name bi-alternants was given to it by Muir. It was Schur who first defined S-function with reference to characters of symmetric group and brought it into the study of group characters. Due to this great contribution, Littlewood suggested the name *Schur-symmetric function* or, *S-function* for brievity. There are several other equivalent definitions for S-functions. These will be given in I.2.2.

Finally, operations on the S-functions and the modification rules of S-functions will be briefly mentioned in I.2.3. The modification rules are the rules to convert non-standard S-functions into standard forms. They will be used extensively in I.4.2 when discussing S-function contents of the new series.

All definitions and formulas here are part of the well established text on the S-function theory. Hence only references are given, proofs will be omitted.

I.2.1 Partitions

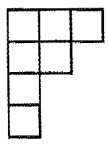
Partitions are important because they are used to label

S-functions. A *partition* (λ) of integer n into p parts, denoted as $\lambda\vdash n$, is a sequence of positive integers $(\lambda)=(\lambda_1,\lambda_2,\ldots,\lambda_p)$ such that $\lambda_1+\lambda_2+\lambda_3+\ldots+\lambda_p=n$. A partition is called *standard* if the sequence $\lambda_1,\ldots,\lambda_p$ is non-increasing, i.e. $\lambda_1\geq\lambda_2\geq,\ldots,\geq\lambda_p>0$. The *weight* of partition (λ) is

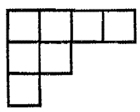
$$\omega(\lambda) = \sum_{i=1}^{p} \lambda_i = n, \qquad (I.2.1\text{-}1)$$

The *length* of partition (λ) is $\ell(\lambda) = \max\{i\,|\,\lambda_i>0\} = p$.

Associated with each partition (λ) there is a frame called a *Young Diagram* (or Ferrers diagram) $F^{(\lambda)}$, consisting of an array of n left adjusted boxes drawn in the plane such that the number of boxes in the i-th row is λ_i. Thus, for example, the Young diagram for $F^{(321^2)}$ is

$F^{(321^2)}$ =  .

Corresponding to each partition (λ), there is a *conjugate partition* denoted by (λ') or $(\lambda)'$ which is obtained by interchanging rows and columns in the Young diagram $F^{(\lambda)}$. For example by interchanging rows and columns of $F^{(321^2)}$ we have

$F^{(321^2)'}$ =  = $F^{(421)}$

thus $(321^2)' = (421)$.

Standard partitions can also be uniquely expressed in the *Frobenius notation* $(\lambda) = \begin{bmatrix} a_1\ldots a_r \\ b_1\ldots b_r \end{bmatrix}$ where r is the number of boxes on the main diagonal of $F^{(\lambda)}$, called the *rank* of partition (λ), a_i, b_i are the number of boxes to the right of and beneath the i-th diagonal box respectively, $a_i > a_{i+1}$, $b_i > b_{i+1}$, for $i=1,2,\dots,r-1$, a_r, $b_r \geq 0$, and

$$r + \sum_{i=1}^{r} (a_i+b_i) = \omega(\lambda). \qquad \text{(I.2.1-2)}$$

Thus partition (321^2) in Frobenius notation is $\left[\begin{smallmatrix}2 & 0\\ 3 & 0\end{smallmatrix}\right]$ which is of rank 2.

Clearly conjugate partitions have a very simple expression in this notation, namely if $(\lambda) = \begin{bmatrix} a_1 \dots a_r \\ b_1 \dots b_r \end{bmatrix}$ then $(\lambda)' = \begin{bmatrix} b_1 \dots b_r \\ a_1 \dots a_r \end{bmatrix}$. This is the reason why sometimes we prefer this notation.

I.2.2 S-functions as Symmetric Functions

The S-function is a special kind of symmetric function. Before its definition is given, we briefly introduce the idea of symmetric functions and some other known symmetric functions which are of importance themselves and also closely related to the S-function .

Let $x_1, x_2, \dots, x_n$ be a set of n indeterminates or variables, n is finite or infinite. A *symmetric (polynomial) function* of x_i's is one which is invariant when the indeterminates are arbitarily permuted. The set of symmetric functions form a graded ring $\Lambda(x_1,\dots,x_n) = \oplus\, \Lambda^k(x_1,\dots,x_n)$ where Λ^k contains symmetric polynomials of homogeneous degree k.

The following symmetric functions are the most common and important (Littlewood 1950, Macdonald 1979):

(i) The *power sum* symmetric function, defined by

$$p_k(x_1,\cdots,x_n) = \sum_{i=1}^{n} x_i^k . \qquad \text{(I.2.2-1)}$$

(ii) The *elementary* symmetric function , defined by

$$e_k(x_1,\cdots,x_n) = \Sigma\, x_1\, x_2,\cdots,x_k \qquad \text{(I.2.2-2a)}$$

where by convention, the summation is assumed to be over all permutations of x_i's. Clearly $e_k = 0$ if $k > n$, $e_o = 1$. The elementary symmetric function corresponding to partition (λ) is defined by

$$e_{(\lambda)} = e_{(\lambda_1,\cdots,\lambda_p)} = e_{\lambda_1}\, e_{\lambda_2}\cdots e_{\lambda_p} \;. \qquad \text{(I.2.2-2b)}$$

(iii) The *homogeneous product sum*, defined by

$$h_k = \sum_{\sigma\vdash k} x_1^{\sigma_1}\, x_2^{\sigma_2},\cdots,x_p^{\sigma_p} \qquad \text{(I.2.2-3a)}$$

summed over all partitions (σ) of weight k, length $p \leq n$ and all permutations of x_i's

$$h_{(\lambda)} = h_{(\lambda_1,\cdots,\lambda_p)} = h_{\lambda_1}\, h_{\lambda_2}\cdots h_{\lambda_p} \;. \qquad \text{(I.2.2-3b)}$$

(iv) The *monomial symmetric function*, defined by

$$m_{(\lambda)}(x_1,\cdots,x_n) = m_{(\lambda_1,\cdots,\lambda_p)} = \Sigma\, x_1^{\lambda_1}\, x_2^{\lambda_2}\cdots x_p^{\lambda_p} \qquad \text{(I.2.2-4)}$$

summed over all permutations of x_i's.

It was shown (Macdonald 1979) that $e_{(\lambda)}$, $h_{(\lambda)}$, $m_{(\lambda)}$, with $\lambda\vdash k$ form integrity bases of Λ^k respectively, i.e. every symmetric function of homogeneous degree k can be expresed as linear combinations of $e_{(\lambda)}$, $h_{(\lambda)}$ or $m_{(\lambda)}$ with integrity coefficients, while $p_{(\lambda)}$ forms a quotient basis of Λ^k.

The S-function $\{\lambda\}$ with $\lambda \vdash k$ is another integrity basis of Λ^k. Classically the S-function is defined as a ratio of two alternants

$$\{\lambda\} = \lim_{n \to \infty} \{\lambda\}\ (x_1, \ldots, x_n), \qquad \text{(I.2.2-5a)}$$

where $$\{\lambda\}(x_1, \ldots, x_n) = |x_i^{\lambda_j+n-j}| / |x_i^{n-j}|, \qquad \text{(I.2.2-5b)}$$

where $|x_i^{n-j}| = \prod_{i<j} (x_i - x_j) = \Delta(x_1, \ldots, x_n)$ is the *alternant*, $i,j = 1,2,\ldots,n$ labels the row and column of matrix.

S-functions can be also expressed as polynomials in other symmetric functions. The equivalent definitions are given below:

(i) Jacobi-Trudi equation

$$\{\lambda\}(x_1, \ldots, x_n) = |h_{\lambda_i - i + j}\ (\underline{x})| \qquad \text{(I.2.2-6a)}$$

specially, $$\{k\} = h_k \qquad \text{(I.2.2-6b)}$$

where i,j labels row and column of the matrix respectively, $h_r(\underline{x})$ is the r-th degree homogeneous product sum, $\underline{x} = x_1, \cdots, x_n$.

(ii) Naegelsbach equation

$$\{\lambda\}(x_1, \ldots, x_n) = |e_{\mu_i - i + j}\ (\underline{x})| \qquad \text{(I.2.2-7a)}$$

specially, $$\{1^k\} = e_k \qquad \text{(I.2.2-7b)}$$

where i,j labels row and column of the matrix respectively, $(\mu) = (\lambda')$ is the conjugate partition of (λ), $e_r(\underline{x})$ is the elementary symmetric function of r-th degree.

(iii) Schur equation

$$\{\lambda\}(x\ , \ldots, x_n) = \frac{1}{n!} \sum_{\rho} h_{\rho}\ \chi_{\rho}^{(\lambda)}\ p_{(\rho)}(\underline{x}) \qquad \text{(I.2.2-8)}$$

where h_{ρ} is the number of elements in the conjugate class (ρ) of the symmetric group S_n, $p_{(\rho)}$ is the power sum symmetric function, $\chi_{\rho}^{(\lambda)}$ is the characteristic of the conjugate class (ρ) in the irreducible representation (λ) of S_n.

(iv) Kostka equation

$$\{\lambda\} = K_{\lambda\mu}\, m_{(\mu)}(\underline{x}) \tag{I.2.2-9}$$

where $(K_{\lambda\mu})$ is the Kostka matrix, $K_{\lambda\mu}$ is the number of tableaux of shape λ and weight μ (Macdonald 1979).

Finally, the S-function is related to the characters of unitary group via the Littlewood equation:

(v) Littlewood equation

$$\{\lambda\}\ (x_1,\dots,x_n) = \chi^{\{\lambda\}}(\phi) \tag{I.2.2-10}$$

where $\chi^{\{\lambda\}}(\phi)$ is the charactertistic of the irreducible representation $\{\lambda\}$ of the unitary group U(n), belonging to the conjugate class labelled by $\phi = (\phi_1,\dots,\phi_n)$ and $x_i = e^{i\phi_i}$, for $i=1,2,\dots,n$ are eigenvalues of the $n \times n$ unitary matrices of class ϕ (Littlewood 1950, Black et al. 1983). This formula provides the crucial link between S-functions and the characters of unitary group and other classical groups.

I.2.3 Modification Rules and S-function Operations

An S-function $\{\lambda\}$ is *non-standard* if (λ) is not a standard partition. To convert a non-standard S-function into a standard one we use the following rules (Littlewood 1950, Wybourne 1970):

Modification rules for S-functions

(i) $\{\lambda_1,\ldots,\lambda_i, \lambda_{i+1},\ldots,\lambda_p\} = -\{\lambda_1,\ldots,\lambda_{i+1}-1, \lambda_i+1,\ldots,\lambda_p\}$

(ii) if $\lambda_i+1 = \lambda_{i+1}$, then $\{\lambda\} = 0$

(iii) Repeat steps (i) and (ii) several times until a standard S-function or zero is reached.

Example:

$$\{3541\} = -\{4441\}$$

$$\{259\} = -\{286\} = \{736\} = -\{754\}$$

$$\{103\} = -\{121\} = 0$$

The modification rules can be proved very easily from the definition of the S-function and the properties of determinants.

S-functions are subject to nine distinct operations as listed below. These operations correspond to operations on the characters of representations of U(n) and on the tensor basis of such representations (King 1975, Wybourne 1970). Properties of these operations have been studied by Littlewood, Wybourne and various other authors.

Operations on S-Functions

$\{\lambda\} + \{\mu\}$	addition
$\{\lambda\} - \{\mu\}$	subtraction
$k\{\lambda\}$	scalar multiplication
$\{\lambda\}\cdot\{\mu\}$ or $\{\lambda\cdot\mu\}$	outer product
$\{\lambda\}\circ\{\mu\}$ or $\{\lambda\circ\mu\}$	inner product ($\omega_\lambda=\omega_\mu$)
$\{\lambda\}/\{\mu\}$ or $\{\lambda/\mu\}$	skew division
$\{\lambda\}\otimes\{\mu\}$ or $\{\mu\}[\{\lambda\}]$	plethysm or wreath product
$\{\lambda\}\odot\{\mu\}$	inner plethysm
$\{\lambda\}'$ or $\{\lambda'\}$	conjugation

Similar operations can also be defined on S-function series. Let $S = \sum_{\sigma} z_{\sigma}\{\sigma\}$ be an S-function series with integer coefficient z_{σ}. We have for example,

$$\{\lambda\}\cdot S = \{\lambda\cdot S\} = \sum_{\sigma} z_{\sigma}\{\lambda\cdot\sigma\} \qquad (I.2.3\text{-}1)$$

$$\{\lambda\}/S = \{\lambda/S\} = \sum_{\sigma} z_{\sigma}\{\lambda/\sigma\} \qquad (I.2.3\text{-}2)$$

$$\{\lambda\}\otimes S = \sum_{\sigma} z_{\sigma}\{\lambda\}\otimes\{\sigma\} \qquad (I.2.3\text{-}3)$$

These operators are very important in the S-function approach to group theory. In fact it is the compact form and simplicity of these operations which makes calculations of group properties such as branching rules and Kronecker products an easy and systematic task.

I.3 CLASSICAL S-FUNCTION SERIES

Having introduced S-functions and outlined their relationship to other symmetric functions and group characters we are in a position to discuss S-function series. This section serves as a review of classical S-function series.

A simple example of a classical S-function series is the L series (so named by King) which is a series of S-functions of the form $\{1^m\}$,

$$L(x): \quad \prod_{i=1}^{\infty} (1-x_i) = \sum_{m=0}^{\infty} (-1)^m\{1^m\} .$$

The left hand side is called the *generating function* of the L series which is a symmetric product function invariant under permutations of the indeterminates x_1, $x_2,\cdots$. The right hand side is the *S-function content* or *expansion* of the series. L series makes its appearance in the unitary group inverse branching rules (King 1975):

$$U(n-1) \uparrow_r U(n), \qquad \{\lambda\} \uparrow_r \{\lambda/L\} \ .$$

Associated with L series there are another three classical S-function series, namely, M, P and Q series, defined by:

$$M: \qquad \prod_{i=1}^{\infty} (1 - x_i)^{-1} = \sum_{m=0}^{\infty} \{m\}$$

$$P: \qquad \prod_{i=1}^{\infty} (1 + x_i)^{-1} = \sum_{m=0}^{\infty} (-1)^m\{m\}$$

$$Q: \qquad \prod_{i=0}^{\infty} (1 + x_i) = \sum_{m=0}^{\infty} \{1^m\}$$

which is the *inverse*, *conjugate* and *adjoint* series of the L series respectively, denoted by:

$$M = L^{-1} \ ,$$

$$P = L' \ ,$$

and $$Q = L^{\dagger} \ .$$

The set of four series related to L will be referred to as the *L-family series*.

Originally there were twelve such series discovered by Littlewood in his work on the characters of classical groups. These series proved important in obtaining branching rules and Kronecker products for compact groups using characters and hence are essential in the S-function approach to group representation theory. Following upon the pioneer work of Littlewood, King (1975) chose to designate the set of distinct infinite series of S-functions by the capital letters A, B, C, D, E, F, G, H, L, M, P and Q which consequently made the symbolic manipulation of S-function series possible. Four other series R, S, W and V were established later (King et al. 1981). These sixteen S-function series with their well defined generating functions and S-function content will be referred to as *classical S-function series*.

In section I.3.1, a table of classical S-function series is given. The sixteen series are grouped into six families. The idea of inverse, conjugate and adjoint series is introduced.

In section I.3.2 the classical S-function series are classified into three types: type I, II and III, according to the form of the generating functions. The inverse, conjugate and adjoint series of each type is summarized. This serves to give an indication as how new S-function series are to be obtained.

Finally in section I.3.3 three other established series X, Y (King et al. 1981) and T (King and Wybourne 1982) are given in terms of their S-function expansions. They are not included in the classical S-function series for the reason that there seems no known generating functions for them.

I.3.1 Brief Review of Classical S-function Series

The sixteen classical S-function series A, B, C, D, E, F, G, H, L, M, P, Q, R, S, V and W with their well defined generating functions and S-function expansions are grouped into six families, namely the A, E, G, L, R and V-family. Each family contains four members (some may be identical): the series itself, the inverse series, the conjugate and the adjoint series which are defined as follows:

(i) *inverse series* S^{-1}: if S is an arbitrary S-function series with generating function $S(x)$, then the inverse series S^{-1} has generating function $S(x)^{-1}$ thus

$$S\,S^{-1} = S^{-1}\,S = 1 \qquad \text{(I.3.1-1)}$$

(ii) *conjugate series* S': if $S = \sum_\lambda z_\lambda \{\lambda\}$ is an arbitrary series of S-functions of the form $\{\lambda\}$, then

$$S' = \sum_\lambda z_\lambda \{\lambda\}' \qquad (I.3.1\text{-}2)$$

i.e. the S-function content of the conjugate series is the conjugate of that of the series.

(iii) *adjoint series* $S^\dagger$: The adjoint series of S is the inverse of the conjugate series of S or, the conjugate of the inverse series of S

$$S^\dagger = (S')^{-1} = (S^{-1})' \qquad (I.3.1\text{-}3)$$

The four properties identity (I), inverse (-1), conjugation (') and adjoint (†) form a discrete four-element group with the multiplication table below:

	I	-1	'	†
I	I	-1	'	†
-1	-1	I	†	'
'	'	†	I	-1
†	†	'	-1	I

In the case of series being self conjugate or self adjoint, namely, $S = S'$ or $S = S^\dagger$, the number of distinct series in a family reduces to two.

The six families of sixteen series with their generating functions and S-function contents are summarized in table (I.3.1).

From observations of the generating functions of the classical S-function series it is clear that the A-family and V-family series can

Table I.3.1 : Sixteen classical S-function series

Family	Series	King's Symbol	Generation function	S-function content	Relationship to the L-family series		Plethysm
L	L	L	$\pi_{i}(1-x_i)$	$\sum_{m=0}^{\infty}(-1)^m\{1^m\}$	$L(x_i)$		$\{1\}\otimes L$
					$L^{\dagger}(-x_i)$		$(-\{1\})\otimes L^{-1}$
	L^{-1}	M	$\pi_{i}(1-x_i)^{-1}$	$\sum_{m=0}^{\infty}\{m\}$	$L^{-1}(x_i)$		$\{1\}\otimes L^{-1}$
					$L'(-x_i)$		$(-\{1\})\otimes L$
	L'	P	$\pi_{i}(1+x_i)^{-1}$	$\sum_{m=0}^{\infty}(-1)^m\{m\}$	$L'(x_i)$		$\{1\}\otimes L'$
					$L^{-1}(-x_i)$		$(-\{1\})\otimes L^{\dagger}$
	$L^{\dagger}$	Q	$\pi_{i}(1+x_i)$	$\sum_{m=0}^{\infty}\{1^m\}$	$L^{\dagger}(x_i)$		$\{1\}\otimes L^{\dagger}$
					$L(-x_i)$		$(-\{1\})\otimes L'$
A	A	A	$\pi_{i<j}(1-x_ix_j)$	$\sum_{\alpha}(-1)^{\omega_\alpha/2}\{\alpha\}$	$L(x_ix_j)$	$i<j$	$\{1^2\}\otimes L$
					$L^{\dagger}(-x_ix_j)$	$i<j$	$(-\{1^2\})\otimes L^{-1}$
	A^{-1}	B	$\pi_{i<j}(1-x_ix_j)^{-1}$	$\sum_{\beta}\{\beta\}$	$L^{-1}(x_ix_j)$	$i<j$	$\{1^2\}\otimes L^{-1}$
					$L'(-x_ix_j)$	$i<j$	$(-\{1^2\})\otimes L$
	A'	C	$\pi_{i\leq j}(1-x_ix_j)$	$\sum_{\gamma}(-)^{\omega_\gamma/2}\{\gamma\}$	$L(x_ix_j)$	$i\leq j$	$\{2\}\otimes L$
					$L^{\dagger}(-x_ix_j)$	$i\leq j$	$(-\{2\})\otimes L^{-1}$
	$A^{\dagger}$	D	$\pi_{i\leq j}(1-x_ix_j)^{-1}$	$\sum_{\delta}\{\delta\}$	$L^{-1}(x_ix_j)$	$i\leq j$	$\{2\}\otimes L^{-1}$
					$L'(-x_ix_j)$	$i\leq j$	$(-\{2\})\otimes L$

Table I.3.1 continued on next page

Table I.3.1 continued

Family	Series	King's Symbol	Generation function	S-function content	Relationship to the L-family series	Plethysm
V	$V=V^{\dagger}$	V	$\prod_i(1-x_i{}^2)$	$\sum_{p,q=0}^{\infty}(-1)^p\{2^p1^{2q}\}$	$L(x_i{}^2)$ $L^{\dagger}(-x_i{}^2)$	$p_2\otimes L$ $(-p_2)\otimes L^{-1}$
	$V^{-1}=V'$	W	$\prod_i(1-x_i{}^2)^{-1}$	$\sum_{p,q=0}^{\infty}(-1)^p\{p+2q,p\}$	$L^{-1}(x_i{}^2)$ $L'(-x_i{}^2)$	$p_2\otimes L^{-1}$ $(-p_2)\otimes L$
E	$E=E'$	E	$\prod_i(1-x_i)\prod_{i<j}(1-x_ix_j)$	$\sum_{\epsilon}(-1)^{(\omega_\epsilon+r)/2}\{\epsilon\}$	$LA=L'A'$	$(\{1\}+\{1^2\})\otimes L$ $(-\{1\}-\{1^2\})\otimes L^{-1}$
	$E^{-1}=E^{\dagger}$	F	$\prod_i(1-x_i)^{-1}\prod_{i<j}(1-x_ix_j)^{-1}$	$\sum_{\zeta}\{\zeta\}$	$L^{-1}A^{-1}=L^{\dagger}A^{\dagger}$	$(\{1\}+\{1^2\})\otimes L^{-1}$ $(-\{1\}-\{1^2\})\otimes L$
G	$G^{-1}=G'$	G	$\prod_i(1+x_i)\prod_{i<j}(1-x_ix_j)$	$\sum_{\epsilon}(-)^{(\omega_\epsilon-r)/2}\{\epsilon\}$	$L^{\dagger}A=L^{-1}A'$	$(-\{1\}+\{2\})\otimes L$ $(\{1\}-\{2\})\otimes L^{-1}$
	$G^{-1}=G^{\dagger}$	H	$\prod_i(1+x_i)^{-1}\prod_{i<j}(1-x_ix_j)^{-1}$	$\sum_{\zeta}(-)^{\omega_\zeta}\{\zeta\}$	$L'A^{-1}=LA^{\dagger}$	$(-\{1\}+\{2\})\otimes L^{-1}$ $(\{1\}-\{2\})\otimes L$
R	$R=R'$	R	$\prod_i(1-x_i)\prod_i(1+x_i)^{-1}$	$\{0\}+2\sum_{a,b=0}^{\infty}(-)^{a+b+1}\left(\begin{smallmatrix}a\\b\end{smallmatrix}\right)$	LL'	
	$R^{-1}=R^{\dagger}$	S	$\prod_i(1+x_i)\prod_i(1-x_i)^{-1}$	$\{0\}+2\sum_{a,b=0}^{\infty}\left(\begin{smallmatrix}a\\b\end{smallmatrix}\right)$	$L^{\dagger}L^{-1}$	

Note: $\{\alpha\}=\begin{bmatrix}a_1\ldots a_r\\ a_1+1\ldots a_r+1\end{bmatrix}$, $\{\gamma\}=\{\alpha\}'$, $\{\delta\}$ has only even parts, $\{\beta\}=\{\delta\}'$, $\{\epsilon\}=\begin{bmatrix}a_1\ldots a_r\\ a_1\ldots a_r\end{bmatrix}$, $\{\zeta\}$ is arbitrary,
$p_2=\sum_i x_i{}^2=\{2\}-\{1^2\}$.

be obtained from L-family series by substitutions which is equivalent to particular plethysms of S-functions (Thomas 1976). Thus under

$$x_i \rightarrow x_i x_j \qquad (i < j),$$

we have

$$\begin{aligned} A &= L(x_i x_j) \\ &= \prod_{i<j} (1 - x_i x_j) \\ &= \{1^2\} \otimes L , \end{aligned} \qquad (I.3.1\text{-}4)$$

and the substitution

$$x_i \rightarrow x_i^2$$

leads to

$$\begin{aligned} V(x_i) &= L(x_i^2) \\ &= \prod_{i=1}^{\infty} (1 - x_i^2) \\ &= p_2 \otimes L , \end{aligned} \qquad (I.3.1\text{-}5)$$

where $p_2 = \sum_i x_i^2 = \{2\} - \{1^2\}$.

The series in other families, E, F and R can be constructed by taking products of the L and A family series. Thus each classical S-function series is obtainable from the L-family series by either substitutions or taking products. The relationship of each series with the L-family series and the relevant plethysms are shown in the last two columns of table I.3.1.

I.3.2 Classification of S-function Series

The six families of classical S-function series are further classified into three types according to their generating functions.

(i) *Type I or S_1-family of S-function series*: The generating function for S_1 involves a single index under the successive product sign $\prod_i$ and has the form:

$$S_1(x_i) = \prod_i (1 - f_1(x_i)) \qquad (I.3.2\text{-}1)$$

where $f_1(x_i)$ is a polynomial function of x_i of degree n, $n \geq 1$.

(ii) *Type II or S_2-family of S-function series* : the generating function of S_2 involves two indices under the successive product sign $\prod_{i<j}$ or $\prod_{i\leq J}$ and has the form:

$$S_2(x_ix_j) = \prod_{i<j\ (i\leq j)} (1 - f_2(x_ix_j)) \qquad (I.3.2\text{-}2)$$

where $f_2(x_ix_j)$ is a polynomial of x_ix_j of degree n, $n \geq 1$.

(iii) *Type III* or *S_3-family of S-function series*: S_3 is a product of two series of type 1 or 2:

$$S_3 = S_iS_j \qquad i,j = 1 \text{ or } 2 \qquad (I.3.2\text{-}3)$$

Classification of the classical S-function series under this scheme is shown in Table I.3.2. The classification is by no means unique, e.g. $V(x_i) = \prod_i(1-x_i^{\,2}) = \prod_i(1-x_i)(1+x_i) = \prod_{i\leq j}(1-x_ix_j)\prod_{i<j}(1-x_ix_j)^{-1}$ can be regarded as either type I or III.

Table I.3.2 : Classification of classical S-function series

Type	Family	Member Series
I	L	L, M, P, Q
	V	V, W (self adjoint)
II	A	A, B, C, D
III	E	E, F (self conjugate)
	G	G, H (self conjugate)
	R	R, S (self conjugate)

Consider the relationship among generating functions of a series and its inverse, conjugate and adjoint series. Firstly, for L, L^{-1}, L' and $L^{\dagger}$ (L-family series, type I), the generating functions satisfy the following relations

$$L^{-1}(x_i) = L(x_i)^{-1} \quad (I.3.2\text{-}4a)$$
$$L'(x_i) = L(-x_i)^{-1} \quad (I.3.2\text{-}4b)$$
$$L^{\dagger}(x_i) = L(-x_i) \quad (I.3.2\text{-}4c)$$

i.e. the adjoint of L is effected by the substitution $x_i \to -x_i$ in L and the conjugation by $x_i \to -x_i$ and $L(x_i) \to L(x_i)^{-1}$.

For A, A^{-1}, A' and $A^{\dagger}$ (A-family series,Type II) we have

$$A^{-1}(x_i x_j) = A\,(x_i x_j)^{-1} \quad (I.3.2\text{-}5a)$$
$$A'(x_i x_j) = \underset{i\leq j}{A}\,(x_i x_j) \quad (I.3.2\text{-}5b)$$
$$A^{\dagger}(x_i x_j) = \underset{i\leq j}{A}\,(x_i x_j)^{-1} \quad (I.3.2\text{-}5c)$$

i.e. the conjugation is effected by the replacement of $\underset{i<j}{\Pi}$ with $\underset{i\leq j}{\Pi}$.

Finally for type III series, e.g. E = LA, we have

$$E^{-1} = L^{-1}A^{-1} \quad (I.3.2\text{-}6a)$$
$$E' = L'\,A' \;(= E) \quad (I.3.2\text{-}6b)$$
$$E^{\dagger} = L^{\dagger}A^{\dagger} \;(= E^{-1}) \quad (I.3.2\text{-}6c)$$

i.e. the conjugate (inverse, adjoint) of a product series is the product of conjugate (inverse, adjoint) series.

These relationships just discussed turn out to be generally true for each type of series (Appendix I). Thus the generating functions for S_i^{-1}, S_i' and $S_i^{\dagger}$ are immediately known once the generating function for S_i

Table I.3.3 : Three Types of S-function Series

Type	Family	Series	Generating function	Relationship to $L(x_i)$
I	S_1	S_1	$\prod_i (1-f_1(x_i))$	$L(f_1(x_i))$ $L^{\dagger}(-f_1(x_i))$
		S_1^{-1}	$\prod_i (1-f_1(x_i))^{-1}$	$L^{-1}(f_1(x_i))$ $L'(-f_1(x_i))$
		S_1'	$\prod_i (1-f_1(-x_i))^{-1}$	$L'(-f_1(-x_i))$ $L^{-1}(f_1(-x_i))$
		$S_1^{\dagger}$	$\prod_i (1-f_1(-x_i))$	$L^{\dagger}(-f_1(-x_i))$ $L(f_1(-x_i))$
II	S_2	S_2	$\prod_{i<j(i\leq j)} (1-f_2(x_ix_j))$	$L(f_2(x_ix_j))$ $i<j$ $(i\leq j)$ $L^{\dagger}(-f_2(x_ix_j))$ $i<j$ $(i\leq j)$
		S_2^{-1}	$\prod_{i<j(i\leq j)} (1-f_2(x_ix_j))^{-1}$	$L^{-1}(f_2(x_ix_j))$ $i<j$ $(i\leq j)$ $L'(-f_2(x_ix_j))$ $i<j$ $(i\leq j)$
		S_2'	$\prod_{i\leq j(i<j)} (1-f_2(x_ix_j))$	$L(f_2(x_ix_j))$ $i\leq j$ $(i<j)$ $L^{\dagger}(-f_2(x_ix_j))$ $i\leq j$ $(i<j)$
		$S_2^{\dagger}$	$\prod_{i\leq j(i<j)} (1-f_2(x_ix_j))^{-1}$	$L^{-1}(f_2(x_ix_j))$ $i\leq j$ $(i<j)$ $L'(-f_2(x_ix_j))$ $i\leq j$ $(i<j)$
III	S_3	S_3	S_iS_j $\quad i,j = 1$ or 2	**Note** : $f_1(x_i), f_2(x_ix_j)$
		S_3^{-1}	$S_i^{-1}S_j^{-1}$	are polynomial functions
		S_3'	$S_i'S_j'$	of x_i, x_ix_j respectively
		$S_3^{\dagger}$	$S_i^{\dagger}S_j^{\dagger}$	of degree $n\geq 1$.

is defined. The results are summarized in table I.3.3.

I.3.3 X, Y, and T Series

The sixteen classical series discussed above have one common feature, i.e. they all have well defined generating functions. In contrast to this, X, Y, T series do not seem to have simple generating functions. As a result of this fact, the inverses of these series are not generally known, but they do exist and can be calculated on a term by term basis. Due to the importance of these series in applications, we list them separately in table I.3.4.

Table I.3.4: **X, Y, T series**

Series	S-function content	
X	$\sum_{\omega'} \{\omega\}'$	$\{\omega\}' = \{2^q 1^{2p}\}$
Y	$\sum_{\omega} \{\omega\}$	$\{\omega\} = \{2p+q \quad q\}$
T	$\sum_{\tau} \{\tau\}$	$\{\tau\} = \{n,n-1,\ldots 2,1\}$

I.4 NEW S-FUNCTION SERIES

In the last section we have classified S-function series into three types. It is clear that there are infinitely many S-function series of each type since $f_1(x_i)$ and $f_2(x_i x_j)$ can be any arbitrary polynomial function. The real difficulty lies in the fact that it is not always possible to get the S-function expansions of these series in general form, although in principle we can always multiply the symmetric products out as monomial symmetric functions and then convert each term into S-functions via the inverse Kostka matrix (Macdonald 1979). However, more than forty new S-functions which have both well defined generating functions as well as S-function expressions have been found. Six of them differ from the classical S-function series essentially by phase factors, but the rest are completely new. The main method employed to obtain the S-function contents of the new series is due to Littlewood (1950), McConnell and Newell (1973).

Generating functions for these new series are tabulated in section I.4.1 and the derivations of their contents are given in section I.4.2.

I.4.1 Generating New S-function Series

The classification scheme discussed in the last section not only gives us a better understanding of the relationships among the classical S-function series but also provides us with a clear insight as to how the new S-function series could be generated. It is clear that the following measures can be adopted:

(i) Make new substitutions for $f_1(x_i)$ in type I series.

(ii) Made new substitutions for $f_2(x_i x_j)$ in type II series.

(iii) Take new possible products.

(iv) Find new types of S-function series.

Method (i) and (ii) can both be regarded as substitutions of new arguments in the L-family series. The substitutional processes are equivalent to some particular plethysms with the L-family series. New series obtained by methods (i) - (iii) always come in groups of four (or two) for as soon as some new generating function is found, its conjugate inverse and adjoint follows immediately from results given in table I.3.3.

The simplest substitution arises in changing the phase of the indeterminates. Under the substitution $x_t \to e^{i\theta}x_t$ $(t=1,2,\dots)$, an S-function $\{\lambda\}$ is changed by a phase:

$$\{\lambda\}(e^{i\theta}x_t) = |(e^{i\theta}x_t)^{n-s+\lambda_s}|/|(e^{i\theta}x_t)^{n-s}|$$

i.e.
$$\{\lambda\}(e^{i\theta}x_t) = (e^{i\theta})^{\omega_\lambda}\{\lambda\}(x_t) \qquad \text{(I.4.1-1)}$$

Particularly

$$\{\lambda\}(i\ x_t) = (-1)^{\omega_\lambda/2}\ \{\lambda\}(x_t) \qquad \text{(I.4.1-2a)}$$

$$\{\lambda\}(-x_t) = (-1)^{\omega_\lambda}\ \{\lambda\}(x_t) \qquad \text{(I.4.1-2b)}$$

$$\{\lambda\}(e^{i\pi/k}x_t) = (-1)^{\omega_\lambda/k}\{\lambda\}(x_t) \qquad \text{(I.4.1-2c)}$$

where ω_λ is the weight of $\{\lambda\}$.

The substitution $x_t \to i\ x_t$ in A and V-family series yields two new families of series denoted by A_+, A_+^{-1}, $A_+{}'$, $A_+{}^\dagger$ and $V_+(=V_+{}^\dagger)$, $V_+{}^{-1}(= V_+{}')$ or, following the original letter designations for classical series by King, as A^+, B^+, C^+, D^+, V^+ and W^+ respectively. Both A_+ and V_+-family series are trivially related to the A and V-family since they differ only by a phase factor $(-1)^{\omega_\lambda/2}$. The generating functions, plethysms and S-function contents of these six new series are summarized in table I.4.1. We observe that similar phase changing in the indeterminates of other classical series do not give rise to any new series.

In table I.4.2 we have listed generating functions and plethysms of a further twelve completely new series which are obtained by the substitution

$$f_1(x_i) = x_i{}^k,\ 1 + x_i + x_i{}^2 +\dots+x_i{}^k \text{ and } 1 + x_i - x_i{}^2 -\dots+(-1)^{k+1}x_i^k$$

respectively, into the L-family series where k is an integer greater than unity.

Table I.4.1 : The A_+ and V_+-family series

Family	Series	King's Symbol	Generating function	S-function content	Substitution	Plethysm
A_+	A_+	A^+	$\prod_{i<j}(1+x_ix_j)$	$\sum_{\alpha}\{\alpha\}$	$L^{\dagger}(x_ix_j)\quad i<j$ $L(-x_ix_j)\quad i<j$	$\{1^2\}\otimes L^{\dagger}$ $(-\{1^2\})\otimes L'$
	A_+^{-1}	B^+	$\prod_{i<j}(1+x_ix_j)^{-1}$	$\sum_{\beta}(-1)^{\omega_\beta/2}\{\beta\}$	$L'(x_ix_j)\quad i<j$ $L^{-1}(-x_ix_j)\quad i<j$	$\{1^2\}\otimes L'$ $(-\{1^2\})\otimes L^{\dagger}$
	A_+'	C^+	$\prod_{i\leq j}(1+x_ix_j)$	$\sum_{\gamma}\{\gamma\}$	$L^{\dagger}(x_ix_j)\quad i\leq j$ $L(-x_ix_j)\quad i\leq j$	$\{2\}\otimes L^{\dagger}$ $(-\{2\})\otimes L'$
	$A_+^{\dagger}$	D^+	$\prod_{i\leq j}(1+x_ix_j)^{-1}$	$\sum_{\delta}(-1)^{\omega_\delta/2}\{\delta\}$	$L'(x_ix_j)\quad i\leq j$ $L^{-1}(-x_ix_j)\quad i\leq j$	$\{2\}\otimes L'$ $(-\{2\})\otimes L^{\dagger}$
V_+	$V_+=V_+^{\dagger}$	V^+	$\prod_{i}(1+x_i^2)$	$\sum_{p,q=0}^{\infty}(-1)^q\{2^p1^{2q}\}$	$L^{\dagger}(x_i^2)$ $L(-x_i^2)$	$p_2\otimes L^{\dagger}$ $(-p_2)\otimes L'$
	$V_+^{-1}=V_+'$	W^+	$\prod_{i}(1+x_i^2)^{-1}$	$\sum_{p,q=0}^{\infty}(-1)^q\{p+2q,p\}$	$L'(x_i^2)$ $L^{-1}(-x_i^2)$	$p_2\otimes L'$ $(-p_2)\otimes L^{\dagger}$

where $\{\alpha\}$, $\{\beta\}$, $\{\gamma\}$ and $\{\delta\}$ are the special S-functions as defined in table I.3.1, $p_2 = \{2\} - \{1^2\}$.

The plethysms for $\pi_i(1+x_i+x_i^2+\cdots+x_i^k)$ and the related series are very interesting. They are derived by using the following identities (Id.1-Id.5) to be proved in Appendix II:

Id.1	$(\sum_\lambda\{\lambda\})\otimes[L]^{-1} = (-\sum_\lambda\{\lambda\})\otimes[L]$	(I.4.1-3a)
Id.2	$(\{\lambda\}\otimes[L])\cdot(\{\mu\}\otimes[L]) = (\{\lambda\}+\{\mu\})\otimes[L]$	(I.4.1-3b)
Id.3	$(\{\lambda\}\otimes[L])' = \begin{cases}\{\lambda\}'\otimes[L]' & \text{if } \omega_\lambda \text{ odd} \\ \{\lambda\}'\otimes[L] & \text{if } \omega_\lambda \text{ even}\end{cases}$	(I.4.1-3c) (I.4.1-3d)
Id.4	$((\sum_\lambda\{\lambda\})\otimes[L])^{-1} = (\sum_\lambda\{\lambda\})\otimes[L]^{-1}$	(I.4.1-3e)
Id.5	$[L](-x_i^k) = -p_k\otimes[L]' = p_k\otimes[L]^\dagger$	(I.4.1-3f)

where $[L] \in \{L, M, P, Q\}$.

Let p_k be the k-th degree power sum symmetric function which is expressible as the sum of S-functions (Macdonald 1979)

$$p_k = \sum_{\substack{\alpha+\beta+1=k \\ \alpha,\beta\geq 0}} (-1)^\beta\{\alpha+1,\ 1^\beta\} = \sum_{\substack{\alpha+\beta+1=k \\ \alpha,\beta\geq 0}} (-1)^\beta \begin{bmatrix}\alpha \\ \beta\end{bmatrix} \tag{I.4.1-4a}$$

then clearly the conjugate of p_k is given by

$$p_{2k+1}' = p_{2k+1} \tag{I.4.1-4b}$$

$$p_{2k}' = -\ p_{2k}\ . \tag{I.4.1-4c}$$

Examples:

$p_1 = \{1\}$	$p_1' = \{1\}$
$p_2 = \{2\}-\{1^2\}$	$p_2' = \{1^2\}-\{2\}$
$p_3 = \{3\}-\{21\}+\{1^3\}$	$p_3' = \{1^3\}-\{21\}+\{3\}$
$p_4 = \{4\}-\{31\}+\{21^2\}-\{1^4\}$	$p_4' = \{1^4\}-\{21^2\}+\{31\}-\{4\}$

It is well known that the substitution $x_i \to x_i^k$ in a series is equivalent to the plethysm of p_k with the series (Macdonald 1979):

$$S(x_i)(x_i^k) = p_k \otimes S(x_i) = S(x_i) \otimes p_k \qquad \text{(I.4.1-5)}$$

Thus we have:

$$\begin{aligned}
&\pi_i(1+x_i+x_i^2+\cdots+x_i^k) \\
&= \pi_i(1-x_i^{k+1})(1-x_i)^{-1} \\
&= L(x_i^{k+1})L^{-1}(x_i) \\
&= (p_{k+1}\otimes L)\cdot(p_1\otimes L^{-1}) && \text{(by (I.4.1-5))} \\
&= (p_{k+1}\otimes L)\cdot(-p_1\otimes L) && \text{(by Id.1)} \\
&= (p_{k+1}-p_1)\otimes L && \text{(by Id.2)} \\
\text{or} \quad &= (-p_{k+1}+p_1)\otimes L^{-1} && \text{(by Id.1)}
\end{aligned}$$

and

$$\begin{aligned}
&\pi_i(1+x_i+x_i^2+\cdots+x_i^k)^{-1} \\
&= ((p_{k+1}-p_1)\otimes L)^{-1} && \text{from the result above} \\
&= (p_{k+1}-p_1)\otimes L^{-1} && \text{(by Id.4)} \\
\text{or} \quad &= (-p_{k+1}+p_1)\otimes L && \text{(by Id.1)}
\end{aligned}$$

also

$$\begin{aligned}
&\pi_i(1-x_i+x_i^2 -\cdots+(-x_i)^k)^{-1} \\
&= (\pi_i(1+x_i+x_i^2 +\cdots+x_i^k))' && \text{(by table I.3.3)} \\
&= ((p_{k+1}-p_1)\otimes L)' \\
&= (p_{k+1}-p_1)\otimes L' \quad \text{if } k \text{ is even} && \text{(by Id.3)} \\
\text{or} \quad &= ((-p_{k+1}+p_1)\otimes L^{-1})' \\
&= (-p_{k+1}+p_1)\otimes L^{\dagger} \quad \text{if } k \text{ is even} && \text{(by Id.3).}
\end{aligned}$$

However, there seems no simple plethysms corresponding to $L(-x_i+x_i^2 -\cdots+(-1)^k x_i^k)$-family series due to the fact that

$(1+x_i-x_i^2+\cdots+(-1)^{k+1}x_i^k)$ does not factorize. The results are summarized in the last column of table I.4.2.

By taking products of L, A and A_+-family series twenty-four interesting new product series appear as shown in table I.4.3(a) and I.4.3(b) where each series in I.4.3(a) is related uniquely to a series in (b) by the phase change in indeterminant:

$$x_i \to -x_i \quad \text{for } i = 1,2,3,\ldots$$

hence their S-function contents will differ only by the phase factor $(-1)^{\omega_\lambda}$. Plethysms of these product series are obtained again by using Id.1-5.

Example:

$$\begin{aligned} LC \ : \quad & \pi_i(1-x_i)\,\pi_{i\leq j}(1-x_ix_j) \\ &= L(x_i)L(x_ix_j) && \text{with} \quad i\leq j \\ &= (\{1\}\otimes L)\cdot(\{2\}\otimes L) \\ &= (\{1\}+\{2\})\otimes L && \text{(by Id.2).} \end{aligned}$$

Finally we list in table I.4.4 further four new series denoted as L^m, L^{-m}, L'^m and $L^{\dagger m}$ or as L^m, M^m, P^m and Q^m which are formed by taking m-th power of the L-family series.

Altogether forty-six new series have been listed in table I.4.1-4. Their S-function contents (apart from those listed in table I.4.1) will be derived in the next section.

I.4.2 S-function Contents of the New Series

There are various ways of obtaining the S-function contents of the new series including directly exploiting S-function products in a product

Table I.4.2 : Twelve new S-function series obtained from L-family series by substitutions

Series	Generating Function	Substitution	Plethysm: k is arbitrary	k is even	k is odd
S	$\pi_i(1-x_i^k)$	$L(x_i^k)$	$p_k\otimes L$		
		$L^\dagger(-x_i^k)$	$(-p_k)\otimes L^{-1}$		
S'	$\pi_i(1-(-x_i)^k)^{-1}$	$L^{-1}((-x_i)^k)$	$((-p_k)\otimes L^{-1})'$	$p_k\otimes L^{-1}$	$-p_k\otimes L^\dagger$
		$L'(-(-x_i)^k)$	$(p_k\otimes L)'$	$-p_k\otimes L$	$p_k\otimes L'$
S^{-1}	$\pi_i(1-x_i^k)^{-1}$	$L^{-1}(x_i^k)$	$p_k\otimes L^{-1}$		
		$L'(-x_i^k)$	$(-p_k)\otimes L$		
$S^\dagger$	$\pi_i(1-(-x_i)^k)$	$L((-x_i)^k)$	$((-p_k)\otimes L)'$	$p_k\otimes L$	$-p_k\otimes L'$
		$L^\dagger(-(-x_i)^k)$	$(p_k\otimes L^{-1})'$	$-p_k\otimes L^{-1}$	$p_k\otimes L^\dagger$
S	$\pi_i(1+x_i+x_i^2+\cdots+x_i^k)$	$L(x_i^{k+1})L'(-x_i)$	$(p_{k+1}-p_1)\otimes L$		
	$=\pi_i(1-x_i^{k+1})(1-x_i)^{-1}$	$L^\dagger(-x_i^{k+1})L^{-1}(x_i)$	$(-p_{k+1}+p_1)\otimes L^{-1}$		
S'	$\pi_i(1-x_i+x_i^2-\cdots+(-x_i)^k)^{-1}$	$L'(-(-x_i)^{k+1})L(-x_i)$	$((p_{k+1}-p_1)\otimes L)'$	$(p_{k+1}-p_1)\otimes L'$	$(-p_{k+1}\otimes L)\cdot(-p_1\otimes L')$
	$=\pi_i(1-(-x_i)^{k+1})^{-1}(1+x_i)$	$L^{-1}((-x_i)^{k+1})L^\dagger(x_i)$	$((-p_{k+1}+p_1)\otimes L^{-1})'$	$(-p_{k+1}+p_1)\otimes L^\dagger$	$(p_{k+1}\otimes L^{-1})\cdot(p_1\otimes L^\dagger)$
S^{-1}	$\pi_i(1+x_i+x_i^2+\cdots+x_i^k)^{-1}$	$L^{-1}(x_i^{k+1})L^\dagger(-x_i)$	$(p_{k+1}-p_1)\otimes L^{-1}$		
	$=\pi_i(1-x_i^{k+1})^{-1}(1-x_i)$	$L'(-x_i^{k+1})L(x_i)$	$(-p_{k+1}+p_1)\otimes L$		
$S^\dagger$	$\pi_i(1-x_i+x_i^2-\cdots+(-x_i)^k)$	$L^\dagger(-(-x_i)^{k+1})L^{-1}(-x_i)$	$((p_{k+1}-p_1)\otimes L^{-1})'$	$(p_{k+1}-p_1)\otimes L^\dagger$	$(-p_{k+1}\otimes L^{-1})\cdot(-p_1\otimes L^\dagger)$
	$=\pi_i(1-(-x_i)^{k+1})(1+x_i)^{-1}$	$L((-x_i)^{k+1}))L'(x_i)$	$((-p_{k+1}+p_1)\otimes L)'$	$(-p_{k+1}+p_1)\otimes L'$	$(p_{k+1}\otimes L)\cdot(p_1\otimes L')$

Table I.4.2 continued on next page

Table I.4.2 continued

Series	Generating Function	Substitution	Plethysm k is arbitrary	k is even	k is odd
S	$\pi_i(1+x_i-x_i^2+\cdots+(-1)^{k+1}x_i^k)$	$L(-x_i+x_i^2-\cdots-(-1)^{k+1}x_i^k)$			
S'	$\pi_i(1-x_i-x_i^2-\cdots-x_i^k)^{-1}$	$L'(-x_i-x_i^2-\cdots-x_i^k)$			
S^{-1}	$\pi_i(1+x_i-x_i^2+\cdots+(-1)^{k+1}x_i^k)^{-1}$	$L^{-1}(-x_i+x_i^2-\cdots-(-1)^{k+1}x_i^k)$			
$S^{\dagger}$	$\pi_i(1-x_i-x_i^2-\cdots-x_i^k)$	$L^{\dagger}(-x_i-x_i^2-\cdots-x_i^k)$			

where $$p_k = \sum_{i=0}^{\infty} x_i^k = \sum_{\substack{\alpha+\beta+1=k\\ \alpha,\beta\geq 0}} (-1)^{\beta}\binom{\alpha}{\beta} = \sum_{\substack{\alpha+\beta+1=k\\ \alpha,\beta\geq 0}} (-1)^{\beta}(\alpha+1,1^{\beta})$$

satisfying $p'_{2k} = -p_{2k}$ $\quad p'_{2k+1} = p_{2k+1}$ (the prime signifies the conjugate S-functions).

Note: There are obviously other equivalent substitutions yielding the same set of series which have not been listed.

Table I.4.3(a) : Twelve new S-function series involving products of L-family series with A and A_+-family series

	Series	King's Symbol	Generating Function	Plethysm
S	$L^{\dagger}A_+$	$E^+ = QA^+$	$\prod_i (1+x_i) \prod_{i<j} (1+x_i x_j)$	$(\{1\}+\{1^2\})\otimes L^{\dagger} = (-\{1\}-\{1^2\})\otimes L'$
S'	$L^{-1}A'_+$	MC^+	$\prod_i (1-x_i)^{-1} \prod_{i\leq j} (1+x_i x_j)$	$((\{1\}+\{1^2\})\otimes L^{\dagger})' = ((-\{1\}-\{1^2\})\otimes L')'$ $= \{1\}\otimes L^{-1} \cdot \{2\}\otimes L^{\dagger}$
S^{-1}	$L'A_+^{-1}$	$F^+ = PB^+$	$\prod_i (1+x_i)^{-1} \prod_{i<j} (1+x_i x_j)^{-1}$	$(\{1\}+\{1^2\})\otimes L' = (-\{1\}-\{1^2\})\otimes L^{\dagger}$
$S^{\dagger}$	$LA_+^{\dagger}$	LD^+	$\prod_i (1-x_i) \prod_{i\leq j} (1+x_i x_j)^{-1}$	$((\{1\}+\{1^2\})\otimes L')' = ((-\{1\}-\{1^2\})\otimes L^{\dagger})'$ $= \{1\}\otimes L \cdot \{2\}\otimes L'$
S	$L^{-1}A_+$	MA^+	$\prod_i (1-x_i)^{-1} \prod_{i<j} (1+x_i x_j)$	$((\{1\}+\{2\})\otimes L^{\dagger})' = ((-\{1\}-\{2\})\otimes L')'$ $= \{1\}\otimes L^{-1} \cdot \{1^2\}\otimes L^{\dagger}$
S'	$L^{\dagger}A'_+$	QC^+	$\prod_i (1+x_i) \prod_{i\leq j} (1+x_i x_j)$	$(\{1\}+\{2\})\otimes L^{\dagger} = (-\{1\}-\{2\})\otimes L'$
S^{-1}	LA_+^{-1}	LB^+	$\prod_i (1-x_i) \prod_{i<j} (1+x_i x_j)^{-1}$	$((\{1\}+\{2\})\otimes L')' = ((-\{1\}-\{2\})\otimes L^{\dagger})'$ $= \{1\}\otimes L \cdot \{1^2\}\otimes L'$
$S^{\dagger}$	$L'A_+^{\dagger}$	PD^+	$\prod_i (1+x_i)^{-1} \prod_{i\leq j} (1+x_i x_j)^{-1}$	$(\{1\}+\{2\})\otimes L' = (-\{1\}-\{2\})\otimes L^{\dagger}$

Table I.4.3(a) continued on next page

Table 1.4.3(a) continued...

Series		King's Symbol	Generating Function	Plethysm
S	$L^{\dagger}A'$	QC	$\pi_{i}(1+x_i)\ \pi_{i\leq j}(1-x_ix_j)$	$((-\{1\}+\{1^2\})\otimes L)'=((\{1\}-\{1^2\})\otimes L^{-1})'$ $=\{1\}\otimes L^{\dagger}\cdot\{2\}\otimes L$
S'	$L^{-1}A$	MA	$\pi_{i}(1-x_i)^{-1}\ \pi_{i<j}(1-x_ix_j)$	$(-\{1\}+\{1^2\})\otimes L=(\{1\}-\{1^2\})\otimes L^{-1}$
S^{-1}	$L'A^{\dagger}$	PD	$\pi_{i}(1+x_i)^{-1}\ \pi_{i\leq j}(1-x_ix_j)^{-1}$	$((-\{1\}+\{1^2\})\otimes L^{-1})'=((\{1\}-\{1^2\})\otimes L)'$ $=\{1\}\otimes L'\cdot\{2\}\otimes L^{-1}$
$S^{\dagger}$	LA^{-1}	LB	$\pi_{i}(1-x_i)\ \pi_{i<j}(1-x_ix_j)^{-1}$	$(-\{1\}+\{1^2\})\otimes L^{-1}=(\{1\}-\{1^2\})\otimes L$

Table I.4.3(b) : Twelve new S-function series involving products of L-family series with A and A_+-family series

Series		King's Symbol	Generating function	Plethysm
S	LA_+	LA^+	$\prod_i(1-x_i)\prod_{i<j}(1+x_ix_j)$	$((-\{1\}+\{2\})\otimes L^\dagger)'=((\{1\}-\{2\})\otimes L')'$
S'	$L'A'_+$	PC^+	$\prod_i(1+x_i)^{-1}\prod_{i\leq j}(1+x_ix_j)$	$(-\{1\}+\{2\})\otimes L^\dagger=(\{1\}-\{2\})\otimes L'$
S^{-1}	$L^{-1}A_+^{-1}$	MB^+	$\prod_i(1-x_i)^{-1}\prod_{i<j}(1+x_ix_j)^{-1}$	$((-\{1\}+\{2\})L')'=((\{1\}-\{2\})\otimes L^\dagger)'$
$S^\dagger$	$L^\dagger A_+^\dagger$	QD^+	$\prod_i(1+x_i)\prod_{i\leq j}(1+x_ix_j)^{-1}$	$(-\{1\}+\{2\})\otimes L'=(\{1\}-\{2\})\otimes L^\dagger$
S	$L'A_+$	PA^+	$\prod_i(1+x_i)^{-1}\prod_{i<j}(1+x_ix_j)$	$(-\{1\}+\{1^2\})\otimes L^\dagger=(\{1\}-\{1^2\})\otimes L'$
S'	LA'_+	LC^+	$\prod_i(1-x_i)\prod_{1\leq j}(1+x_ix_j)$	$((-\{1\}+\{1^2\})\otimes L^\dagger)'=((\{1\}-\{1^2\})\otimes L')'$
S^{-1}	$L^\dagger A_+^{-1}$	QB^+	$\prod_i(1+x_i)\prod_{1<j}(1+x_ix_j)^{-1}$	$(-\{1\}+\{1^2\})\otimes L'=(\{1\}-\{1^2\})\otimes L^\dagger$
$S^\dagger$	$L^{-1}A_+^\dagger$	MD^+	$\prod_i(1-x_i)^{-1}\prod'_{i\leq j}(1+x_ix_j)^{-1}$	$((-\{1\}+\{1^2\})\otimes L')'=((\{1\}-\{1^2\})\otimes L^\dagger)'$

Table I.4.3(b) continued on next page

Table I.4.3(b) continued..

Series		King's Symbol	Generating function	Plethysm
S	LA'	LC	$\prod_i(1-x_i)\prod_{i\leq j}(1-x_ix_j)$	$(\{1\}+\{2\})\otimes L=(-\{1\}-\{2\})\otimes L^{-1}$
S'	L'A	PA	$\prod_i(1+x_i)^{-1}\prod_{i<j}(1-x_ix_j)$	$((\{1\}+\{2\})\otimes L)'=((-\{1\}-\{2\})\otimes L^{-1})'$
S^{-1}	$L^{-1}A^{\dagger}$	MD	$\prod_i(1-x_i)^{-1}\prod_{i\leq j}(1-x_ix_j)^{-1}$	$(\{1\}+\{2\})\otimes L^{-1}=(-\{1\}-\{2\})\otimes L$
$S^{\dagger}$	$L^{\dagger}A^{-1}$	QB	$\prod_i(1+x_i)\prod_{i<j}(1-x_ix_j)^{-1}$	$((\{1\}+\{2\})\otimes L^{-1})'=((-\{1\}-\{2\})\otimes L)'$

Note: Each series in table I.4.3(b) is related to a series in table I.4.3(a) by the phase change in the indeterminants: $x_i \to -x_i$ for $i=1,2,\cdots$.

Table I.4.4 : L^m-family series

	Series	Generating function	Relationship to L-family series	Plethysm
S	L^m	$\underset{i}{\pi}(1-x_i)^m$	$(L(x_i))^m$	$(m\{1\})\otimes L$
			$(L^\dagger(-x_i))^m$	$(-m\{1\})\otimes L^{-1}$
S^{-1}	L^{-m}	$\underset{i}{\pi}(1-x_i)^{-m}$	$(L^{-1}(x_i))^m$	$(m\{1\})\otimes L^{-1}$
	(M^m)		$(L'(-x_i))^m$	$(-m\{1\})\otimes L$
S'	L'^m	$\underset{i}{\pi}(1+x_i)^{-m}$	$(L'(x_i))^m$	$(m\{1\})\otimes L'$
	(P^m)		$(L^{-1}(-x_i))^m$	$(-m\{1\})\otimes L^\dagger$
$S^\dagger$	$L^{\dagger m}$	$\underset{i}{\pi}(1+x_i)^m$	$(L^\dagger(x_i))^m$	$(m\{1\})\otimes L^\dagger$
	(Q^m)		$(L(-x_i))^m$	$(-m\{1\})\otimes L'$

Note: $m>0$ is a positive integer.

series, e.g. for PB^+ and LD^+ series and using known S-function formulas and identities, e.g. for L^m-family series. However, the most important method which will be used to derive most of the new series is the determinant method, first given by Littlewood in his derivation of A and C series and later on developed and sharpened by McConnell and Newell (1973). The principal idea involved is to try to write down the completely symmetric generating function as a ratio of a determinant or generalized Vandermonde determinant over the alternant, identify S-function content by using properties of determinants and the definition of S-function and then convert non-standard S-functions into standard forms systematically using the modification rules. We outline the method as follows. To find S-function content of a series generated by $S(x_i)$ or $S(x_ix_j)$:

(i) multiply the generating function by the alternant

$$\Delta(x_i) = \prod_{i<j} (x_i - x_j) = |x_t^{n-s}| \qquad \text{(I.4.2-1)}$$

where t,s labels the rows and columns of the n-th order determinant.

(ii) Rearrange the antisymmetric product $\Delta(x_i)S$ either as some determinant (if $S=S(x_i)$), or as a product of symmetric function with generalized Vandermonde determinant (if $S=S(x_ix_j)$):

$$\begin{aligned}\Delta(x_i)\ S(x_ix_j) &= \prod_{i=1}^{n} g(x_i) \prod_{i<j} (\theta(x_i) - \theta(x_j)) \\ &= \prod_{i=1}^{n} g(x_i)\ |\theta(x_t)^{n-s}| \\ &= |g(x_t)\ \theta(x_t)^{n-s}| \qquad \text{(I.4.2-2)}\end{aligned}$$

(iii) Use the following properties (prop.1-3) of n × n determinant $|a_{ts}|$ (with $t,s=1,2,\cdots,n$).

prop.1 $\prod_{i=1}^{n} c_i |a_{ts}| = |c_s a_{ts}| = |c_t a_{ts}|$

prop.2 $|a_{ts}| = |a_{t1} \cdots a_{ti} \cdots a_{tn}|$

$$= |a_{t1} \cdots a_{ti} + \sum_{k=1}^{n} c_k a_{tk} \cdots a_{tn}|$$

prop.3 $|a_{t1} \cdots a_{ti} + b_{ti} \cdots a_{tn}|$

$$= |a_{t1} \cdots a_{ti} \cdots a_{tn}| + |a_{t1} \cdots b_{ti} \cdots a_{tn}|$$

to simplify the determinant found in (ii) down to a sum of determinates.

(iv) Divide the expression thus obtained by the alternant $\Delta(x_i)=|x_t^{n-s}|$ and identify the non-standard S-function content of the series by the definition of S-functions (I.2.2-5). Let the number of indeterminate n become infinite, i.e. $n \to \infty$.

(v) Modify the non-standard S-functions by applying the modification rules given in section I.2.3 to achieve the standard S-functions and calculate the multiplicities.

The results are tabulated as follows:

Table I.4.5 gives the non-standard S-function contents of the twelve new series listed in table I.4.2, namely the $\pi_i(1+x_i^k)$, $\pi_i(1+x_i+x_i^2+\cdots+x_i^k)$ and $\pi_i(1+x_i-x_i^2+\cdots-(-x_i)^k)$-family series.

Table I.4.6-11 gives the S-function contents both non-standard and standard for the above series in the special cases when k=3 or 4, namely, for $\pi_i(1+x_i^3)$, $\pi_i(1+x_i^4)$, $\pi_i(1+x_i+x_i^2)$, $\pi_i(1+x_i-x_i^2)$, $\pi_i(1+x_i+x_i+x_i^3)$ and $LV = \pi_i(1-x_i-x_i^2+x_i^3)$-family series respectively.

Table I.4.12 and 13 gives the non-standard and standard S-function contents for the twelve new product series listed in table I.4.3(a). S-function contents for the series in table I.4.3(b) can be obtained from these results by multiplying the phase factor $(-1)^{\omega_\lambda}$. Hence they are not listed separately.

Table I.4.5 : Non-standard S-function contents for the new series listed in Table I.4.2

Series	Generating function	Non-standard S-function expansion (summation is over all non-standard $\{\lambda\}$)	Non-standard S-function $\{\lambda\}$
S	$\prod_i(1+x_i^k)$	$\sum_\lambda \begin{Bmatrix} 0 & 0\cdots 0\cdots \\ k & k\cdots k\cdots \end{Bmatrix}$	$\lambda_s=0$ or k for $s=1,2,\ldots$
S^{-1}	$\prod_i(1+x_i^k)^{-1}$	$\sum_\lambda (-1)^{\omega_\lambda/k}\{m_1k\ m_2k\cdots m_kk\}$ with $m_i\geq 0$	$\lambda_s=m_sk$ with $m_s\geq 0$ for $s=1,2,\cdots,k$.
S	$\prod_i(1-x_i^k)$	$\sum_\lambda (-1)^{\omega_\lambda/k}\begin{Bmatrix} 0 & 0\cdots 0\cdots \\ k & k\cdots k\cdots \end{Bmatrix}$	$\lambda_s=0$ or k for $s=1,2,\cdots$.
S^{-1}	$\prod_i(1-x_i^k)^{-1}$	$\sum_\lambda \{m_1k\ m_2k\cdots m_kk\}$ with $m_i\geq 0$	$\lambda_s=m_sk$ with $m_s\geq 0$ for $s=1,2,\cdots,k$.
S	$\prod_i(1+x_i+x_i^2+\cdots+x_i^k)$	$\sum_\lambda \begin{Bmatrix} 0 & 0\cdots 0\cdots \\ 1 & 1\cdots 1\cdots \\ 2 & 2\cdots 2\cdots \\ \vdots & \vdots \quad \vdots \\ k & k\cdots k\cdots \end{Bmatrix}$	$\lambda_s=0,1,2,\cdots$ or k for $s=1,2,\cdots$.
S'	$\prod_i(1-x_i+x_i^2-\cdots+(-x_i)^k)^{-1}$	$\sum_\lambda (-1)^{\omega_\lambda}\begin{Bmatrix} m_1(k+1) & m_2(k+1) & \cdots & m_k(k+1) \\ -(m_1(k+1)+1) & -(m_2(k+1)+1) & \cdots & -(m_k(k+1)+1) \end{Bmatrix}$	$\lambda_s=m_s(k+1)$ or $-(m_s(k+1)+1)$ for $s=1,2,\cdots,k$.

Table I.4.5 continued on next page

Table I.4.5 : Non-standard S-function contents for the new series listed in Table I.4.2

Series	Generating function	Non-standard S-function expansion (summation is over all non-standard $\{\lambda\}$)	Non-standard S-function $\{\lambda\}$
S	$\prod_i (1+x_i^{k})$	$\sum_\lambda \begin{Bmatrix} 0 & 0\cdots 0\cdots \\ k & k\cdots k\cdots \end{Bmatrix}$	$\lambda_s=0$ or k for $s=1,2,\ldots$
S^{-1}	$\prod_i (1+x_i^{k})^{-1}$	$\sum_\lambda (-1)^{\omega_\lambda/k} \{m_1k\ m_2k\cdots m_kk\}$ with $m_i\geq 0$	$\lambda_s=m_sk$ with $m_s\geq 0$ for $s=1,2,\cdots,k.$
S	$\prod_i (1-x_i^{k})$	$\sum_\lambda (-1)^{\omega_\lambda/k} \begin{Bmatrix} 0 & 0\cdots 0\cdots \\ k & k\cdots k\cdots \end{Bmatrix}$	$\lambda_s=0$ or k for $s=1,2,\cdots.$
S^{-1}	$\prod_i (1-x_i^{k})^{-1}$	$\sum_\lambda \{m_1k\ m_2k\cdots m_kk\}$ with $m_i\geq 0$	$\lambda_s=m_sk$ with $m_s\geq 0$ for $s=1,2,\cdots,k.$
S	$\prod_i (1+x_i+x_i^{2}+\cdots+x_i^{k})$	$\sum_\lambda \begin{Bmatrix} 0 & 0\cdots 0\cdots \\ 1 & 1\cdots 1\cdots \\ 2 & 2\cdots 2\cdots \\ \vdots & \vdots \quad \vdots \\ k & k\cdots k\cdots \end{Bmatrix}$	$\lambda_s=0,1,2,\cdots$ or k for $s= 1,2,\cdots.$
S'	$\prod_i (1-x_i+x_i^{2}-\cdots+(-x_i)^{k})^{-1}$	$\sum_\lambda (-1)^{\omega_\lambda} \begin{Bmatrix} m_1(k+1) & m_2(k+1) & \cdots & m_k(k+1) \\ -(m_1(k+1)+1) & -(m_2(k+1)+1) & \cdots & -(m_k(k+1)+1) \end{Bmatrix}$	$\lambda_s=m_s(k+1)$ or $-(m_s(k+1)+1)$ for $s=1,2,\cdots,k.$

Table I.4.5 continued on next page

Table I.4.5 continued

Series	Generating function	Non-standard S-function expansion (summation is over all non-standard $\{\lambda\}$)	Non-standard S-function $\{\lambda\}$
$S^{\dagger}$	$\prod_i(1-x_i+x_i^2-\cdots+(-x_i)^k)$	$\Sigma(-1)^{\omega_\lambda}\left\{\begin{matrix} 0 & 0\cdots 0\cdots \\ 1 & 1\cdots 1\cdots \\ 2 & 2\cdots 2\cdots \\ \vdots & \vdots \quad \vdots \\ k & k\cdots k\cdots \end{matrix}\right\}$	$\lambda_s=0,1,2,\cdots$ or k for $s=1,2,\cdots$.
S^{-1}	$\prod_i(1+x_i+x_i^2+\cdots+x_i^k)^{-1}$	$\sum_\lambda\left\{\begin{matrix} m_1(k+1) & m_2(k+1) & \cdots & m_k(k+1) \\ -(m_1(k+1)+1) & -(m_2(k+1)+1) & \cdots & -(m_k(k+1)+1) \end{matrix}\right\}$	$\lambda_s=m_s(k+1)$ or $-(m_s(k+1)+1)$ for $s=1,2,\cdots,k$.
S	$\prod_i(1+x_i-x_i^2+\cdots+(-1)^{k+1}x_i^k)$	$\sum_\lambda\left\{\begin{matrix} 0 & 0 & \cdots & 0 & \cdots \\ 1 & 1 & \cdots & 1 & \cdots \\ -2 & -2 & \cdots & -2 & \cdots \\ \vdots & \vdots & & \vdots & \\ (-1)^{k+1}k & (-1)^{k+1}k & & (-1)^{k+1}k & \cdots \end{matrix}\right\}$	$\lambda_s=0,1,-2,\cdots$ or $(-1)^{k+1}k$ for $s=1,2,\cdots$.
S'	$\prod_i(1-x_i-x_i^2-\cdots-x_i^k)^{-1}$		
$S^{\dagger}$	$\prod_i(1-x_i-x_i^2-\cdots-x_i^k)$	$\sum_\lambda\left\{\begin{matrix} 0 & 0\cdots & 0\cdots \\ -1 & -1\cdots & -1\cdots \\ -2 & -2\cdots & -2\cdots \\ \vdots & \vdots & \vdots \\ -k & -k\cdots & -k\cdots \end{matrix}\right\}$	$\lambda_s=0,-1,-2,\cdots$ or $-k$ for $s=1,2,\cdots$.
S^{-1}	$\prod_i(1+x_i-x_i^2+\cdots+(-1)^{k+1}x_i^k)^{-1}$		

Table I.4.6 : S-function contents for $\pi_i(1+x_i^3)$ and the related new series

Series	Generating functions	Non-standard S-function expansion for the series	Standard S-function expansion for the series
S	$\underset{i}{\pi}(1+x_i^3)$	$\sum\limits_{\lambda}\left\{\begin{matrix}0 & 0\cdots0\cdots\\ 3 & 3\cdots3\cdots\end{matrix}\right\}$	$\sum\limits_{n,p,q=0}^{\infty}(\{3^n2^{3p}1^{3q}\}-\{3^n2^{3p+1}1^{3q+1}\})$
S'	$\underset{i}{\pi}(1-x_i^3)^{-1}$	$\sum\limits_{\lambda}\{3m_1\ 3m_2\ 3m_3\}$	$\sum\limits_{m;s\geq t=0}^{\infty}(\{m+3s\ m+3t\ m\}-\{m+3s+2\ m+3t+1\ m\})$
$S^{\dagger}$	$\underset{i}{\pi}(1-x_i^3)$	$\sum\limits_{\lambda}(-1)^{\omega_\lambda/3}\left\{\begin{matrix}0 & 0\cdots0\cdots\\ 3 & 3\cdots3\cdots\end{matrix}\right\}$	$\sum\limits_{n,p,q=0}^{\infty}(-1)^{n+q}(\{3^n2^{3p}1^{3q}\}+\{3^n2^{3p+1}1^{3q+1}\})$
S^{-1}	$\underset{i}{\pi}(1+x_i^3)^{-1}$	$\sum\limits_{\lambda}(-1)^{\omega_\lambda/3}\{3m_1\ 3m_2\ 3m_3\}$	$\sum\limits_{m;s\geq t=0}^{\infty}(-1)^{m+s+t}(\{m+3s\ m+3t\ m\}+\{m+3s+2\ m+3t+1\ m\})$

where $\lambda_s=0$ or 3 for $s=1,2,\ldots,$ in S and $S^{\dagger}$. $\lambda_s=3m_s$ for $s=1,2,3$ with $m_s\geq0$ in S', $S^{\dagger}$.

Table I.4.7 : S-function contents for $\pi_i(1+x_i^4)$ and the related new series

Series	Generating function	Non-standard S-function expansion for the series	Standard S-function expansion for the series
$S=S^\dagger$	$\pi_i(1+x_i^4)$	$\sum_\lambda \begin{Bmatrix}0 & 0\cdots0\cdots\\ 4 & 4\cdots4\cdots\end{Bmatrix}$	$\sum_{m,n,p,q=0}^{\infty}(-1)^{n+q}(\{4^m3^{4n}2^{4p}1^{4q}\}+\{4^m3^{4n+2}2^{4p+1}1^{4q}\}+\{4^m3^{4n}2^{4p+1}1^{4q+2}\}$ $+\{4^m3^{4n+2}2^{4p+2}1^{4q+2}\}-\{4^m3^{4n+1}2^{2p}1^{4q+1}\})$
$S^{-1}=S'$	$\pi_i(1+x_i^4)^{-1}$	$\sum_\lambda(-1)^{\omega_\lambda/4}\{4m_1\ 4m_2\ 4m_3\ 4m_4\}$	$\sum_{m,p;r\geq s\geq t=0}^{\infty}((-1)^{r-s+t}(\{m+4r\ m+4s\ m+4t\ m\}+\{m+4r+3\ m+4s+3\ m+4t+2\ m\}$ $+\{m+4r+3\ m+4s+1\ m+4t\ m\}+\{m+4r+6\ m+4s+4\ m+4t+2\ m\})$ $-(-1)^r\ \{m+4r+2p+2\ m+4s+2p+1\ m+4s+1\ m\})$
$S=S^\dagger$	$\pi_i(1-x_i^4)$	$\sum_\lambda(-1)^{\omega_\lambda/4}\begin{Bmatrix}0 & 0\cdots0\cdots\\ 4 & 4\cdots4\cdots\end{Bmatrix}$	$\sum_{m,n,p,q=0}^{\infty}(-1)^{m}(\{4^m3^{4n}2^{4p}1^{4q}\}+\{4^m3^{4n+2}2^{4p+1}1^{4q}\}-\{4^m3^{4n}2^{4p+1}1^{4q+2}\}$ $-\{4^m3^{4n+2}2^{4p+2}1^{4q+2}\}+(-1)^q\{4^m3^{4n+1}2^{2p}1^{4q+1}\})$
$S^{-1}=S'$	$\pi_i(1-x_i^4)^{-1}$	$\sum_\lambda\{4m_1\ 4m_2\ 4m_3\ 4m_4\}$	$\sum_{m,p;r\geq s\geq t=0}^{\infty}(-1)^{m}(\{m+4r\ m+4s\ m+4t\ m\}+\{m+4r+3\ m+4s+3\ m+4t+2\ m\}$ $-\{m+4r+3\ m+4s+1\ m+4t\ m\}-\{m+4r+6\ m+4s+4\ m+4t+2\ m\}$ $+(-1)^{m+p}\{m+4r+2p+2\ m+4s+2p+1\ m+4s+1\ m\})$

where $\lambda_s=0$ or 4 for $s=1,2\ldots$, in $S(=S^\dagger)$, $\lambda_s=4m_s$ for $s=1,2,3,4$ with $m_s\geq0$ in S^{-1} $(=S')$.

Table I.4.8 : S-function contents for $\prod_i (1+x_i+x_i^2)$ and the related new series

Series	Generating function	Non-standard S-function expansion for the series	Standard S-function expansion for the series
S	$\prod_i (1+x_i+x_i^2)$	$\sum_\lambda \begin{Bmatrix} 0 & 0\cdots 0\cdots \\ 1 & 1\cdots 1\cdots \\ 2 & 2\cdots 2\cdots \end{Bmatrix}$	$\sum_{p,q=0}^{\infty} \phi(q)\ \{2^p 1^q\}$
S'	$\prod_i (1-x_i+x_i^2)^{-1}$	$\sum_\lambda (-1)^{\omega_\lambda} \begin{Bmatrix} 3m_1 & 3m_2 & 3m_3 \\ -(3m_1+1) & -(3m_2+1) & -(3m_3+1) \end{Bmatrix}$	$\sum_{p,q=0}^{\infty} \phi(q)\ \{p+q\ p\}$
$S^\dagger$	$\prod_i (1-x_i+x_i^2)$	$\sum_\lambda (-1)^{\omega_\lambda} \begin{Bmatrix} 0 & 0\cdots 0\cdots \\ 1 & 1\cdots 1\cdots \\ 2 & 2\cdots 2\cdots \end{Bmatrix}$	$\sum_{p,q=0}^{\infty} \theta(q)\ \{2^p 1^q\}$
S^{-1}	$\prod_i (1+x_i+x_i^2)^{-1}$	$\sum_\lambda \begin{Bmatrix} 3m_1 & 3m_2 & 3m_3 \\ -(3m_1+1) & -(3m_2+1) & -(3m_3+1) \end{Bmatrix}$	$\sum_{p,q=0}^{\infty} \theta(q)\ \{p+q\ p\}$

where

$$\phi(q) = \sum_{\gamma=0}^{[q/2]} (-1)^x \begin{bmatrix} q-x \\ x \end{bmatrix} = \begin{cases} 1 & \text{if } q=0,1 \bmod 6, \\ 0 & \quad\ 2,5 \bmod 6, \\ -1 & \quad\ 3,4 \bmod 6. \end{cases} \quad \text{and} \quad \theta(q) = (-1)^q \phi(q) = \begin{cases} 1 & \text{if } q=0 \bmod 3, \\ 0 & \quad\ 2 \bmod 3, \\ -1 & \quad\ 1 \bmod 3. \end{cases}$$

$[q/2]$ is the integer part of $q/2$, $\begin{bmatrix} n \\ k \end{bmatrix} = \dfrac{n!}{k!(n-k)!}$ is the binomial coefficient.

Table I.4.9 : S-Function contents for $\prod_i(1+x_i-x_i^2)$ and the related new series

Series	Generating function	Non-standard S-function expansion	Standard S-function expansion
S	$\prod_i(1+x_i-x_i^2)$	$\sum_\lambda \begin{Bmatrix} 0 & 0\cdots0\cdots \\ 1 & 1\cdots1\cdots \\ -2 & -2\cdots2\cdots \end{Bmatrix}$	$\sum_{p,q=0}^{\infty} (-1)^p\psi(q)\{2^p1^q\}$
S'	$\prod_i(1-x_i-x_i^2)^{-1}$		$\sum_{p,q=0}^{\infty} (-1)^p\psi(q)\{p+q,p\}$
$S^\dagger$	$\prod_i(1-x_i-x_i^2)$	$\sum_\lambda(-1)^{\omega_\lambda} \begin{Bmatrix} 0 & 0\cdots & 0\cdots \\ 1 & 1\cdots & 1\cdots \\ -2 & -2\cdots & -2\cdots \end{Bmatrix}$	$\sum_{p,q=0}^{\infty} (-1)^{p+q}\psi(q)\{2^p1^q\}$
S^{-1}	$\prod_i(1+x_i-x_i^2)^{-1}$		$\sum_{p,q=0}^{\infty} (-1)^{p+q}\psi(q)\{p+q,p\}$

where $\psi(q) = \sum_{x=0}^{[q/2]} \begin{bmatrix} q-x \\ x \end{bmatrix}$, [q/2] is the integer part of q/2,

$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{n!}{k!(n-k)!}$ is the binomial coefficient.

Table I.4.10 : S-function contents for $\pi_i(1+x_i+x_i^2+x_i^3)$ and the related new series

Series		Generating function	Non-standard S-function expansion	Standard S-function expansion
S	$L^{\dagger}V_+$ (QV^+)	$\pi_i(1+x_i+x_i^2+x_i^3)$	$\sum_{\lambda} \left\{\begin{matrix} 0 & 0\cdots 0\cdots \\ 1 & 1\cdots 1\cdots \\ 2 & 2\cdots 2\cdots \\ 3 & 3\cdots 3\cdots \end{matrix}\right\}$	$\{0\} + \sum_{m,n,p=0}^{\infty} \phi(n)\psi(p)(\{3^m2^n1^p\}-\{3^m2^{n+1}1^{p+1}\})$
$S^{\dagger}$	LV_+ (LV^+)	$\pi_i(1-x_i+x_i^2-x_i^3)$	$\sum_{\lambda}(-1)^{\omega_\lambda} \left\{\begin{matrix} 0 & 0\cdots 0\cdots \\ 1 & 1\cdots 1\cdots \\ 2 & 2\cdots 2\cdots \\ 3 & 3\cdots 3\cdots \end{matrix}\right\}$	$\{0\} + \sum_{m,n,p=0}^{\infty} (-1)^m\phi(n)\theta(p)(\{3^m2^n1^p\}+\{3^m2^{n+1}1^{p+1}\})$
S'	$L^{-1}V_+$ (MW^+)	$\pi_i(1-x_i+x_i^2-x_i^3)^{-1}$	$\sum_{\lambda}(-1)^{\omega_\lambda} \left\{\begin{matrix} 4m_1 & 4m_2 & 4m_3 \\ -4(m_1+1) & -4(m_2+1) & -4(m_3+1) \end{matrix}\right\}$	$\{0\} + \sum_{r\geq s\geq t=0}^{\infty} \phi(s-t)\psi(r-s)(\{r\ s\ t\}-\{r+2\ s+1\ t\})$
S^{-1}	$L'V_+^{-1}$ (PW^+)	$\pi_i(1+x_i+x_i^2+x_i^3)^{-1}$	$\sum_{\lambda} \left\{\begin{matrix} 4m_1 & 4m_2 & 4m_3 \\ -4(m_1+1) & -4(m_2+1) & -4(m_3+1) \end{matrix}\right\}$	$\{0\} + \sum_{r\geq s\geq t=0}^{\infty} (-1)^t\phi(s-t)\theta(r-s)(\{r\ s\ t\}+\{r+2\ s+1\ t\})$

where
$$\phi(n) = \sum_{z=0}^{[(n-3y)/2]} \sum_{y=0}^{[n/3]} (-1)^z \frac{(n-2y-z)!}{y!z!(n-3y-2z)!} = \begin{cases} 1 & \text{if } n=0,1 \mod 4 \\ 0 & \phantom{\text{if } n=0,}2,3 \mod 4 \end{cases}$$

$$\psi(p) = \sum_{z=0}^{[(p-3y)/2]} \sum_{y=0}^{[p/3]} (-1)^z \frac{(p-2y-z)!}{y!z!(p-3y-2z)!} = \begin{cases} 1 & \text{if } p=0,1 \mod 4 \\ 0 & \phantom{\text{if } p=0,}2,3 \mod 4 \end{cases}$$

$$\theta(p) = (-1)^p\psi(p) = \begin{cases} 1 & \text{if } p = 0 \mod 4 \\ 0 & \phantom{\text{if } p =\ } 2,3 \mod 4 \\ -1 & \phantom{\text{if } p = } 1 \mod 4 \end{cases}$$

[n] is the integer part of n

Table I.4.11 : S-function for $\pi_i(1-x_i-x_i^2+x_i^3)$ and the related new series

Series		Generating function	Non-standard S-function expansion	Standard S-function expansion
S	LV	$\pi_i(1-x_i-x_i^2+x_i^3)$	$\sum_\lambda \left\{\begin{matrix} 0 & 0\cdots & 0\cdots \\ -1 & -1\cdots\cdot & -1\cdots \\ -2 & -2\cdots\cdot & -2\cdots \\ 3 & 3\cdots & 3\cdots \end{matrix}\right\}$	$\{0\} + \sum_{m,n,p=0}^{\infty} \Omega(n)\tau(p)(\{3^m2^n1^p\} - \{3^m2^{n+1}1^{p+1}\})$
$S^\dagger$	$L^\dagger V^\dagger$ (QV)	$\pi_i(1+x_i-x_i^2-x_i^3)$	$\sum_\lambda (-1)^{\omega_\lambda} \left\{\begin{matrix} 0 & 0\cdots & 0\cdots \\ -1 & -1\cdots\cdot & -1\cdots \\ -2 & -2\cdots\cdot & -2\cdots \\ 3 & 3\cdots & 3\cdots \end{matrix}\right\}$	$\{0\} + \sum_{m,n,p=0}^{\infty} (-1)^m\Omega(n)\theta(p)\{3^m2^n1^p\} + \{3^m2^{n+1}1^{p+1}\})$
S'	L'V' (PW)	$\pi_i(1+x_i-x_i^2-x_i^3)^{-1}$		$\{0\} + \sum_{r\geq s\geq t=0}^{\infty} \Omega(s-t)\tau(r-s)(\{r\ s\ t\} - \{r+2\ s+1\ t\})$
S^{-1}	$L^{-1}V^{-1}$ (MW)	$\pi_i(1-x_i-x_i^2+x_i^3)^{-1}$		$\{0\} + \sum_{r\geq s\geq t=0}^{\infty} (-1)^m\Omega(s-t)\theta(r-s)(\{r\ s\ t\} + \{r+2\ s+1\ t\})$

where $$\Omega(n) = (-1)^n \sum_{z=0}^{[(n-3y)/2]} \sum_{y=0}^{[n/3]} (-1)^y \frac{(n-2y-z)!}{y!z!(n-3y-2y)!} = (-)^n([n/2]+1)$$

$$\tau(p) = (-)^p \sum_{z=0}^{[(p-3y)/2]} \sum_{y=0}^{[p/3]} (-1)^y \frac{(p-2y-z)!}{y!z!(p-3y-2y)!} = (-)^p([p/2]+1)$$

and $$\theta(p) = (-1)^p\,\tau(p) = \sum_{z=0}^{[(p-3y)/2]} \sum_{y=0}^{[p/3]} (-1)^y \frac{(p-2y-z)!}{y!z!(p-3y-2y)!} = [p/2]+1$$

[n] is the integer part of n

Table I.4.12 : Non-standard S-function contents for the new product series listed in Table I.4.3(a)

Series		Generating function	Non-standard S-function expansion for the series (summation is over all non-standard S-function $\{\lambda\}$)	Non-standard S-function $\{\lambda\}$
S	$L^{\dagger}A_{+}$ (QA^{+}) (E^{+})	$\pi_{i}(1+x_i)\ \pi_{i<j}(1+x_ix_j)$	$\sum_{\lambda}\left\{\begin{array}{cccccc} 0 & 0 & 0 & 0\cdots & 0 & \cdots \\ 1 & 1 & 1 & 1\cdots & 1 & \cdots \\ & -2 & 4 & -6\cdots & (-1)^{s+1}(2s-2) & \cdots \\ & -3 & 5 & -7\cdots & (-1)^{s+1}(2s-1) & \cdots \end{array}\right\}$	$\lambda_1 = 0$ or 1 $\lambda_s = 0, 1, (-1)^{s+1}(2s-2)$ or $(-1)^{s+1}(2s-1)$ for $s = 2, 3, 4, \cdots$.
S'	$L^{-1}A'_{+}$ (MC^{+})	$\pi_{i}(1-x_i)^{-1}\ \pi_{i\leq j}(1+x_ix_j)$	$\sum_{\lambda}\left\{\begin{array}{cccccc} 0 & 0 & 0 & 0 & 0\cdots & 0 & \cdots \\ 1 & 1 & 1 & 1 & 1\cdots & 1 & \cdots \\ & 2 & -4 & 6 & -8\cdots & (-1)^{s}(2s-2) & \cdots \\ & 3 & -5 & 7 & -9\cdots & (-1)^{s}(2s-1) & \cdots \end{array}\right\}$ $+2\sum_{\lambda}\left\{\begin{array}{cccccc} m & 0 & 0 & 0 & 0\cdots & 0 & \cdots \\ & 1 & 1 & 1 & 1\cdots & 1 & \cdots \\ & 2 & -4 & 6 & -8\cdots & (-1)^{s}(2s-2) & \cdots \\ & 3 & -5 & 7 & -9\cdots & (-1)^{s}(2s-1) & \cdots \end{array}\right\}$	$\lambda_1 = 0, 1$ or $m(\geq 2)$ $\lambda_s = 0, 1, (-1)^{s}(2s-2)$ or $(-1)^{s}(2s-1)$ for $s = 2, 3, 4. \cdots$.
S^{-1}	$L'A_{+}^{-1}$ (PB^{+})	$\pi_{i}(1+x_i)^{-1}\ \pi_{i<j}(1+x_ix_j)^{-1}$	$\sum_{\lambda}(-1)^{\omega_\beta/2}\{(-1)^{m_1}(\beta_1+m_1)\ (-1)^{m_2}(\beta_2+m_2)\cdots (-1)^{m_{2\ell}}(\beta_{2\ell}+m_{2\ell})\ (-1)^{m_{2\ell+1}}m_{2\ell+1}\cdots\}$	$\lambda_s = \begin{cases} (-1)^{m_s}(\beta_s+m_s) \\ \text{for } s = 1, 2, \cdots, 2\ell \\ (-1)^{m_s}\, m_s \text{ for } s > 2\ell \end{cases}$ with $m_s \geq 0$ for $s \geq 1$, $\ell \geq 0$.
$S^{\dagger}$ $(S^{-1})'$	$LA_{+}^{\dagger}$ (LD^{+})	$\pi_{i}(1-x_i)\ \pi_{i\leq j}(1+x_ix_j)^{-1}$	$\sum_{\lambda}(-1)^{\omega_\delta/2}\left\{\begin{array}{cccccc} \delta_1 & \delta_2 & \cdots & \delta_\ell & 0\cdots & 0\cdots \\ -(\delta_1+1) & -(\delta_2+1) & \cdots & -(\delta_\ell+1) & -1\cdots & -1\cdots \end{array}\right\}$	$\lambda_s = \begin{cases} \delta_s \text{ or } -(\sigma_s+1) \\ \text{for } s = 1, 2, \cdots, \ell \\ 0 \text{ or } -1 \text{ for } s > \ell \end{cases}$ with $\ell \geq 0$

Table I.4.12 continued on next page

Table I.4.12 continued

	Series	Generating function	Non-standard S-function expansion for the series (summation is over all non-standard S-function $\{\lambda\}$)	Non-standard S-function $\{\lambda\}$
S	$L^{-1}A_+$ (MA^+)	$\prod_i(1-x_i)^{-1}\prod_{i<j}(1+x_ix_j)$	$\sum_\lambda \left\{\begin{array}{llllll} m & 0 & 0\ 0 & 0\cdots & 0 & \cdots \\ & 1 & 1\ 1 & 1 & 1 & \cdots \\ & & -2\ 4 & -6\cdots & (-1)^s(2s-4) & \cdots \\ & & -3\ 5 & -7\cdots & (-1)^s(2s-3) & \cdots \end{array}\right\}$	$\lambda_1 = m\ (\geq 0)$ $\lambda_2 = 0$ or 1 $\lambda_s = 0, 1, (-1)^s(2s-4)$ or $(-1)^s(2s-3)$ for $s = 3,4,5,\cdots$.
S'	$L^\dagger A'_+$ (QC^+)	$\prod_i(1+x_i)\prod_{i\leq j}(1+x_ix_j)$	$\sum_\lambda \left\{\begin{array}{llllll} 0 & 0 & 0 & 0\cdots & 0 & \cdots \\ 1 & 1 & 1 & 1\cdots & 1 & \cdots \\ 2 & -4 & 6 & -8\cdots & (-1)^{s+1}2s & \cdots \\ 3 & -5 & 7 & -9\cdots & (-1)^{s+1}(2s+1) & \cdots \end{array}\right\}$	$\lambda_s = 0, 1, (-1)^{s+1}$ or $(-1)^{s+1}(2s+1)$ for $s = 1,2,3,\cdots$.
S^{-1}	LA_+^{-1} (LB^+)	$\prod_i(1-x_i)\prod_{i<j}(1+x_ix_j)^{-1}$	$\sum_\lambda(-1)^{\omega_\beta/2}\left\{\begin{array}{llllll} \beta_1 & \beta_2 & \cdots & \beta_{2\ell} & 0\cdots & 0\cdots \\ -(\beta_1+1) & -(\beta_2+1) & \cdots & -(\beta_{2\ell}+1) & -1\cdots & -1\cdots \end{array}\right\}$	$\lambda_s = \begin{cases} \beta_s \text{ or } -(\beta_s+1) \\ \text{for } s = 1,2,\ldots,2\ell \\ 0 \text{ or } -1 \text{ for } s > 2\ell \end{cases}$ with $\ell \geq 0$.
$S^\dagger$ $(S^{-1})'$	$L'A_+^\dagger$ (PD^+)	$\prod_i(1+x_i)^{-1}\prod_{i\leq j}(1+x_ix_j)^{-1}$	$\sum_\lambda(-1)^{\omega_\delta/2}\{(-1)^{m_1}(\delta_1+m_1)\ (-1)^{m_2}(\delta_2+m_2)\cdots$ $(-1)^{m_\ell}(\delta_\ell+m_\ell)\ (-1)^{m_{\ell+1}}m_{\ell+1}\cdots\}$	$\lambda_s = (-1)^{m_s}(\delta_s+m_s)$ for $s = 1,2,\cdots,\ell$ $(-1)^{m_s}m_s$ for $s > \ell$ with $\ell \geq 0, m_s \geq 0$.

Table I.4.12 continued on next page

Table I.4.12 continued

	Series	Generating function	Non-standard S-function expansion for the series (summation is over all non-standard S-function $\{\lambda\}$)	Non-standard S-function $\{\lambda\}$
S	$L^{\dagger}A'$ (QC)	$\prod_i (1+x_i) \prod_{i\leq j} (1-x_ix_j)$	$\sum_\lambda \left\{\begin{matrix} 0 & 0 & 0 & 0\cdots & 0 & \cdots \\ 1 & 1 & 1 & 1\cdots & 1 & \cdots \\ -2 & -4 & -6 & -8\cdots & -2s & \cdots \\ -3 & -5 & -7 & -9\cdots & -(2s+1) & \cdots \end{matrix}\right\}$	$\lambda_s = 0, 1, -2s$ or $-(2s+1)$ for $s = 1,2,3,\cdots$.
S'	$L^{\dagger}A$ (MA)	$\prod_i (1-x_i)^{-1} \prod_{i<j} (1-x_ix_j)$	$\sum_\lambda \left\{\begin{matrix} m & 0 & 0 & 0\cdots & 0 & \cdots \\ & -1 & -3 & -5\cdots & -(2s-3) & \cdots \end{matrix}\right\}$	$\lambda_1 = m\ (\geq 0)$, $\lambda_s = 0$, or $-(2s-3)$ for $s = 2,3,\cdots$.
S^{-1}	$L'A^{\dagger}$ (PD)	$\prod_i (1+x_i)^{-1} \prod_{i\leq j} (1-x_ix_j)^{-1}$	$\sum_\lambda \{(-1)^{m_1}(m_1+\delta_1)\ (-1)^{m_2}(\delta_2+m_2)\cdots (-1)^{m_\ell}(\delta_\ell+m_\ell)\ (-1)^{m_{\ell+1}} m_{\ell+1}\cdots\}$	$\lambda_s = \begin{cases} (-1)^{m_s}(\delta_s+m_s) \\ \text{for } s = 1,2,\cdots\ell, \\ (-1)^{m_s}\, m_s \text{ for } s > \ell. \end{cases}$
$S^{\dagger}$ $(S^{-1})'$	LA^{-1} (LB)	$\prod_i (1-x_i) \prod_{i<j} (1-x_ix_j)^{-1}$	$\sum_\lambda \left\{\begin{matrix} \beta_1 & \beta_2 & \cdots & \beta_{2\ell} & 0\cdots & 0\cdots \\ -(\beta_1+1) & -(\beta_2+1)\cdots & & -(\beta_{2\ell+1}) & -1\cdots & -1\cdots \end{matrix}\right\}$	$\lambda_s = \begin{cases} \beta_s \text{ or } -(\beta_s+1) \\ \text{for } s = 1,2,\cdots,2\ell, \\ 0 \text{ or } -1 \text{ for } s > 2\ell \end{cases}$ with $\ell \geq 0$.

Note: $B = \sum_\beta \{\beta\}$ with $\beta_1 = \beta_2 \geq \beta_3 = \beta_4 \geq \cdots \geq \beta_{2\ell-1} = \beta_{2\ell}$ for $\ell \geq 0$, $\omega_\beta = \beta_1 + \beta_2 + \cdots \beta_{2\ell}$.

$D = \sum_\sigma \{\delta\}$ where $\delta_1 \geq \delta_2 \geq \cdots \geq \delta_\ell > 0$ are even integers for $\ell \geq 0$, $\omega_\delta = \delta_1+\delta_2+\cdots+\delta_\ell$.

Table I.4.13 : Standard S-function contents for the new product series listed in Table I.4.3(a)

	Series	Generating function	Standard S-function expansion for the series
S	L^+A_+ $(QA^+)(E^+)$	$\prod_i (1+x_i) \prod_{i<j} (1+x_ix_j)$	$\{0\} + \sum^{\infty}_{\substack{a_1>a_2>\cdots>a_r=0 \\ b_1>b_2>\cdots>b_r=0}} \sum^{\infty}_{r=1} \phi(r;\underline{a},\underline{b}) \begin{bmatrix} a_1a_2\cdots a_r \\ b_1b_2\cdots b_r \end{bmatrix}$
S'	$L^{-1}A'_+$ (MC^+)	$\prod_i (1-x_i)^{-1} \prod_{i\leq j} (1+x_ix_j)$	$\{0\} + \sum^{\infty}_{\substack{a_1>a_2>\cdots>a_r=0 \\ b_1>b_2>\cdots>b_r=0}} \sum^{\infty}_{r=1} \phi(r;\underline{a},\underline{b}) \begin{bmatrix} b_1b_2\cdots b_r \\ a_1a_2\cdots a_r \end{bmatrix}$
	where $\phi(r;\underline{a},\underline{b}) = \prod_{i=1}^{r} \text{mult} \begin{bmatrix} a_i \\ b_i \end{bmatrix} = \text{mult} \begin{bmatrix} a_1 \\ b_1 \end{bmatrix} \cdot \text{mult} \begin{bmatrix} a_2 \\ b_2 \end{bmatrix} \cdots \text{mult} \begin{bmatrix} a_r \\ b_r \end{bmatrix}$ with $\text{mult} \begin{bmatrix} a_i \\ b_i \end{bmatrix} = \begin{cases} 0 & \text{if } a_i>b_i \text{ or } a_i<b_{i+1} \\ 1 & a_i=b_i \text{ or } a_i=b_{i+1} \\ 2 & a_i<b_i \text{ and } a_i>b_{i+1} \end{cases}$ for $i=1,2,\cdots,r$.		
S^{-1}	$L'A_+^{-1}$ $(PB^+)(F^+)$	$\prod_i (1+x_i)^{-1} \prod_{i<j} (1+x_ix_j)^{-1}$	$\sum_\zeta (-1)^{(\omega_\zeta+n_c)/2} \{\zeta\}$
$S^\dagger$	$LA_+^\dagger$ (LD^+)	$\prod_i (1-x_i) \prod_{i\leq j} (1+x_ij_i)^{-1}$	$\sum_\zeta (-1)^{(\omega_\zeta+n_r)/2} \{\zeta\}$
	where $\{\zeta\}$ is an arbitrary S-function of weight ω_ζ. n_r, n_c is the number of odd rows and columns of $\{\zeta\}$ respectively.		

Table I.4.13 continued on next page

Table I.4.13 continued

	Series	Generating function	Standard S-function expansion for the series
S	$L^{-1}A_+$ (MA^+)	$\pi_{i}(1-x_i)^{-1} \pi_{i<j}(1+x_ix_j)$	$\{0\} + \sum\limits^{\infty}_{\substack{a_1>a_2>\cdots>a_r=0 \\ b_1>b_2>\cdots>b_r=0}} \sum\limits^{\infty}_{r=1} \psi(r;\underline{a},\underline{b}) \begin{bmatrix} a_1a_2\cdots a_r \\ b_1b_2\cdots b_r \end{bmatrix}$
S'	$L^{\dagger}A'_+$ (QC^+)	$\pi_{i}(1+x_i)\, \pi_{i\leq j}(1+x_ix_j)$	$\{0\} + \sum\limits^{\infty}_{\substack{a_1>a_2>\cdots>a_r=0 \\ b_1>b_2>\cdots>b_r=0}} \sum\limits^{\infty}_{r=1} \psi(r;\underline{a},\underline{b}) \begin{bmatrix} b_1b_2\cdots b_r \\ a_1a_2\cdots a_r \end{bmatrix}$

where $\psi(r;\underline{a},\underline{b}) = \text{mult} \begin{bmatrix} a_1a_2\cdots a_r \\ b_1b_2\cdots b_r \end{bmatrix} = \text{mult} \begin{bmatrix} a_1 \\ b_1 \end{bmatrix} \ \text{mult} \begin{bmatrix} a_2 \\ b_2 \end{bmatrix} \cdots \text{mult} \begin{bmatrix} a_r \\ b_r \end{bmatrix}$

with $\text{mult} \begin{bmatrix} a_i \\ b_i \end{bmatrix} = \begin{cases} 0 & \text{if } a_i>b_i+2 \text{ or } a_i<b_{i+1}+2\ , \\ 1 & \phantom{\text{if }} a_i=b_i+2 \text{ or } a_i=b_{i+1}+2\ , \\ 2 & \phantom{\text{if }} a_i<b_i+2 \text{ and } a_i>b_{i+1}+2\ . \end{cases}$ $\text{mult} \begin{bmatrix} a_r \\ b_r \end{bmatrix} = \begin{cases} 0 & \text{if } a_r>b_r+2, \\ 1 & \phantom{\text{if }} a_r=b_r+2 \text{ or, if } a_r = 0 \text{ or } 1, \\ 2 & \phantom{\text{if }} a_r<b_r+2\ . \end{cases}$

for $i=1,2,\cdots,r-1$.

Table I.4.13 continued on next page

Table 1.4.13 continued

	Series	Generating function	Standard S-function expansion for the series
S^{-1}	LA_+^{-1} (LB^+)	$\prod_i (1-x_i) \prod_{i<j} (1+x_ix_j)^{-1}$	$\{0\} + \sum_{\underline{k},\underline{k}',\underline{\epsilon},\lambda,n} (-1)^{m+n}\{\lambda_1^{4k_1}(\lambda_1\lambda_1-1)^{\epsilon_1}(\lambda_1-1)^{4k'_1}\cdots\lambda_\ell^{4k_\ell}(\lambda_\ell\lambda_\ell-1)^{\epsilon_\ell}(\lambda_\ell-1)^{4k'_\ell}1^n\}$
	where $\underline{k}=k_1,k_2,\cdots,k_\ell \geq 0$, $\underline{k}'=k'_1,k'_2,\cdots,k'_\ell \geq 0$, $\underline{\epsilon}=\epsilon_1,\epsilon_2,\cdots,\epsilon_\ell = 0$ or 1 only, and $m = \sum_{i=1}^{\ell} \epsilon_i\lambda_i$. $\{\lambda\} = \{\lambda_1,\lambda_2,\cdots,\lambda_\ell\}$ with $\lambda_1>\lambda_2>\cdots>\lambda_\ell > 1$, $n=0$ or 1 mod 4		
$S^\dagger$	$L'A_+^\dagger$ (PD^+)	$\prod_i (1+x_i)^{-1} \prod_{i\leq j} (1+x_ix_j)^{-1}$	$\sum_\nu (-1)^{(\omega_\nu+n_o)/2}\{\nu\}$
	where $\nu_i-\nu_{i+1} = \begin{cases} 0,\ 1 \bmod 4 & \text{if } \nu_i \text{ is even,} \\ 1,\ 2 \bmod 4 & \text{if } \nu_i \text{ is odd,} \end{cases}$ and n_o is the number of odd rows in $\{\nu\}$.		

Table I.4.13 continued on next page

Table I.4.13 continued

	Series	Generating function	Standard S-function expansion for the series
S	$L^{\dagger}A$ (QC)	$\pi_{i}(1+x_i)\ \pi_{i\leq j}(1-x_ix_j)$	$\{0\} + \sum_{a_1>a_2>\cdots>a_r=0}^{\infty}\ \sum_{b_1>b_2>\cdots>b_r=0}^{\infty}\ \sum_{r=1}^{\infty} \Lambda(r;\underline{a},\underline{b}) \begin{bmatrix} a_1a_2\cdots a_r \\ b_1b_2\cdots b_r \end{bmatrix}$
S'	$L^{-1}A$ (MA)	$\pi_{i}(1-x_i)^{-1}\ \pi_{i<j}(1-x_ix_j)$	$\{0\} + \sum_{a_1>a_2>\cdots>a_r=0}^{\infty}\ \sum_{b_1>b_2>\cdots>b_r=0}^{\infty}\ \sum_{r=1}^{\infty} \Lambda(r;\underline{a},\underline{b}) \begin{bmatrix} b_1b_2\cdots b_r \\ a_1a_2\cdots a_r \end{bmatrix}$

where $\Lambda(r;\underline{a},\underline{b}) = \text{mult}\begin{bmatrix} a_1a_2\cdots a_r \\ b_1b_2\cdots b_r \end{bmatrix} = \text{mult}\begin{bmatrix} a_1 \\ b_2 \end{bmatrix}\ \text{mult}\begin{bmatrix} a_2 \\ b_2 \end{bmatrix}\ \cdots\ \text{mult}\begin{bmatrix} a_r \\ b_r \end{bmatrix}$

with $\text{mult}\begin{bmatrix} a_i \\ b_i \end{bmatrix} = \begin{cases} (-1)^{a_r-1} & \text{if } a_i=b_i+2 \text{ (and } a_i>b_{i+1}+2\text{)}, \\ (-1)^{a_r} & (a_i<b_i+2 \text{ and) } a_i=b_{i+1}+2, \\ 0 & a_i>b_i+2 \text{ or } a_i<b_{i+1}+2. \end{cases}$ $\quad \text{mult}\begin{bmatrix} a_r \\ b_r \end{bmatrix} = \begin{cases} 0 & \text{if } a_r \neq b_r+2 \\ 1 & a_r = 0 \\ -1 & a_r = 1 \\ (-1)^{a_r-1} & a_r = b_r+2 \end{cases}$

for $i=1,2,\cdots,r-1$

Note: conditions inside the brackets are necessarily satisfied since it is assumed that $a_1>a_2>\cdots>a_r$ and $b_1>b_2>\cdots>b_r$.

Table I.4.13 continued on next page

Table I.4.13 continued

	Series	Generating function	Standard S-function expansion for the series
S^{-1}	$L'A^{\dagger}$ (PD)	$\prod_i (1+x_i)^{-1} \prod_{i\leq j} (1-x_i x_j)^{-1}$	
$S^{\dagger}$	LA^{-1} (LB)	$\prod_i (1-x_i) \prod_{i<j} (1-x_i x_j)^{-1}$	$\{0\} + \sum_{\underline{k},\underline{k}',\underline{\epsilon},\lambda,n} (-1)^{\epsilon_1+\epsilon_2+\cdots+\epsilon_\ell+n}\, \Omega(n;\underline{k},\underline{k}')\{\lambda_1^{2k_1}(\lambda_1\lambda_1-1)^{\epsilon_1}(\lambda_1-1)^{2k'_1}\cdots \lambda_\ell^{2k_\ell}(\lambda_\ell\lambda_\ell-1)^{\epsilon_\ell}(\lambda_\ell-1)^{2k'_\ell}1^n\}$

where $\underline{k}=k_1,k_2,\cdots,k_\ell \geq 0$, $\underline{k}'=k'_1,k'_2,\cdots,k'_\ell \geq 0$, $\underline{\epsilon}=\epsilon_1,\epsilon_2,\cdots,\epsilon_\ell = 0$ or 1,
$\{\lambda\} = \{\lambda_1\lambda_2\cdots\lambda_\ell\}$ with $\lambda_1>\lambda_2>\cdots>\lambda_\ell \geq 2$,

and $\Omega(n;\underline{k},\underline{k}') = \prod_{i=1}^{\ell} (k_i+1)\,(k_i'+1)\,[(n+2)/2]$, $[(n+2)/2]$ is the integer part of $(n+2)/2$.

If $\lambda_\ell = 2$, it is assumed $k_\ell' = 0$.

Table I.4.14 : Standard S-function contents for L^m-family series

Series	Generating function	Standard S-function expansion
L^m	$\pi_i(1-x_i)^m$	$\sum_\lambda (-1)^{\omega_\lambda} \phi_m^{\lambda'} \{\lambda\}$
$L'^m = P^m$	$\pi_i(1+x_i)^{-m}$	$\sum_\lambda (-1)^{\omega_\lambda} \phi_m^{\lambda} \{\lambda\}$
$L^{\dagger m} = Q^m$	$\pi_i(1+x_i)^m$	$\sum_\lambda \phi_m^{\lambda'} \{\lambda\}$
$L^{-1m} = M^m$	$\pi_i(1-x_i)^{-m}$	$\sum_\lambda \phi_m^{\lambda} \{\lambda\}$

where $\phi_m^\lambda = f^{(\lambda)}/n!$ times the product of the first λ_i terms from

$$\begin{array}{lllll} m, & m+1, & m+2, & m+3, & \cdots \\ m-1, & m, & m+1, & m+2, & \cdots \\ m-2, & m-1, & m, & m+1, & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdots \end{array}$$

$f^{(\lambda)}$ is the degree of the corresponding character of the symmetric group S_n, where $\lambda \vdash n$.

Finally, table I.4.14 gives the standard S-function contents for the L^m-family series listed in table I.4.4.

Statements and proofs of some typical results from each table will be given. Other proofs may be shown similarly.

Results from table I.4.5:

Statement 1 : $\displaystyle\pi_i(1+x_i^k) = \sum_\lambda \begin{Bmatrix} 0 & 0\cdots 0\cdots \\ k & k\cdots k\cdots \end{Bmatrix}$

summed over all non-standard S-function $\{\lambda\}$ in which λ_s=0 or k for s=1,2,$\cdots$.

Proof 1 : Following the determinant method described at the beginning of this section,

$$\begin{aligned}
&\pi_{i=1}^{n} (1+x_i^k) \\
&= |x_t^{n-s}| \pi_{i=1}^{n} (1+x_i^k)\Delta(x_i)^{-1} \\
&= |x_t^{n-s} - x_t^{n-s+k}|\Delta(x_i)^{-1} && \text{(by prop.1)} \\
&= \sum_{s=1}^{n} \sum_{\lambda_s=0 \text{ or } k} |x_t^{n-s+\lambda_s}|/|x_t^{n-s}| && \text{(by prop.3)} \\
&= \sum_\lambda \{\lambda_1\ \lambda_2\cdots\lambda_n\} && \text{(by I.2.2-5)}
\end{aligned}$$

where λ_s = 0 or k for s=1,2,$\cdots$,n. Let $n \to \infty$ to yield

$$\pi_i(1+x_i^k) = \sum_\lambda \begin{Bmatrix} 0 & 0\cdots 0\cdots \\ k & k\cdots k\cdots \end{Bmatrix}$$

Statement 2 : $\displaystyle\pi_i(1-x_i^k)^{-1} = \sum_\lambda\{m_1k\ m_2k\cdots m_kk\}$

summed over all non-standard S-function $\{\lambda\}$ in which $\lambda_s=m_sk$ for s=1,2,$\cdots$,k. (λ_s=0 for s>k).

Proof 2 :

$$\prod_{i=1}^{n} (1-x_i^{k})^{-1}$$

$$= |x_t^{n-s}| \prod_{i=1}^{n} (1-x_i^{k})^{-1} \Delta(x_i)^{-1}$$

$$= \left|\frac{x_t^{n-s}}{1-x_t^{k}}\right| \Delta(x_i)^{-1} \qquad \text{(by prop.1)}$$

$$= \left|\frac{x_t^{n-1}}{1-x_t^{k}} \ \frac{x_t^{n-2}}{1-x_t^{k}} \cdots \frac{x_t^{n-k}}{1-x_t^{k}} \ \frac{x_t^{n-k-1}}{1-x_t^{k}} \cdots \frac{x_t^{n-n}}{1-x_t^{k}}\right| \Delta(x_i)^{-1}.$$

Apply the elementary operations on columns c(s) for $k<s\leq n$ to yield new columns c'(s) :

$$c(s) - c(s-k) \to c'(s) = x_t^{n-s}.$$

Expand $\frac{1}{1-x_t^{k}}$ out as $\frac{1}{1-x_t^{k}} = \sum_{m_s=0}^{\infty} x_i^{m_s k}$

Thus

$$\prod_{i=1} (1-x_i^{k})^{-1}$$

$$=\left|\sum_{m_1=0}^{\infty} x_t^{n-1+m_1 k}, \ \sum_{m_2=0}^{\infty} x_t^{n-2+m_2 k}, \cdots \sum_{m_k=0}^{\infty} x_t^{n-k+m_k k}, x_t^{n-k-1}, \cdots, x_t^{n-n}\right| \Delta(x_i)^{-1}$$

(by prop.2)

$$= \sum_{\lambda_s} |x_t^{n-s+\lambda_s}| / |x_t^{n-s}| \qquad \text{(by prop.3)}$$

$$= \sum_{\lambda} \{\lambda_1 \ \lambda_2 \cdots \lambda_k \cdots \lambda_n\} \qquad \text{(by (I.2.2-5))}$$

where $\lambda_s = m_s k$ for $s=1,2,\cdots,k$ and $\lambda_s=0$ for $k<s\leq n$. Let $n \to \infty$

$$\prod_i (1-x_i^{k})^{-1} = \sum_{\lambda} \{m_1 k \ m_2 k \cdots m_k k\}.$$

Statement 3 : $$\prod_i (1-x_i^{k}) = \sum_{\lambda} (-1)^{\omega_\lambda / k} \begin{Bmatrix} 0 & 0 \cdots 0 \cdots \\ k & k \cdots k \cdots \end{Bmatrix}$$

summed over all non-standard S-function $\{\lambda\}$ of weight ω_λ in which $\lambda_s=0$ or k for $s=1,2,\cdots$.

Proof 3 : Consider the series in variables $x_t = x_1, x_2, \cdots$ written as $\prod_i(1+x_i{}^k) = \sum_\lambda\{\lambda\}(x_t)$ and the phase change in the indeterminates

$$x_t \to e^{i\pi/k}\, x_t .$$

We obtain :

$$\begin{aligned}
&\prod_t(1-x_t{}^k) \\
&= \prod_t(1+(e^{i\pi/k}\, x_t)^k) \\
&= \sum_\lambda\{\lambda\}(e^{i\pi/k}\, x_t) \\
&= \sum_\lambda(-1)^{\omega_\lambda/k}\,\{\lambda\}\,(x_t) && \text{(by (I.4.1-2c))} \\
&= \sum_\lambda(-1)^{\omega_\lambda/k}\begin{Bmatrix} 0 & 0\cdots0\cdots \\ k & k\cdots k\cdots \end{Bmatrix} . && \text{(by Statement 1)}
\end{aligned}$$

i.e. the S-function content for $\prod_i(1-x_i{}^k)$ differs from $\prod_i(1+x_i{}^k)$ by the phase factor $(-1)^{\omega_\lambda/k}$ only.

Results from table I.4.6 :

Statement 4 : The standard S-function content for $\prod_i(1+x_i{}^3)$ series is

$$\prod_i(1+x_i{}^3) = \sum_{n,p,q=0}^{\infty}(\{3^n2^{3p}1^{3q}\} - \{3^n2^{3p+1}1^{3q+1}\}).$$

Proof 4 : The non-standard S-function expansion for the series is (from table I.4.5, k=3)

$$\begin{aligned}
\prod_i(1+x_i{}^3) &= \sum_\lambda\begin{Bmatrix} 0 & 0\cdots0\cdots \\ 3 & 3\cdots3\cdots \end{Bmatrix} \\
&= \{0\} + \sum_{k=1}^{\infty}\sum_{i=1}^{k-1}\sum_{\lambda_i=0 \text{ or } 3}\{\lambda_1\lambda_2\cdots\lambda_{k-1}\ 3\}
\end{aligned}$$

$$= \{0\} + \{3\} + \sum_{\lambda_1=0 \text{ or } 3} \{\lambda_1 3\} + \sum_{i=1}^{2} \sum_{\lambda_i=0 \text{ or } 3} \{\lambda_1\lambda_2\, 3\}$$

$$+ \sum_{i=1}^{3} \sum_{\lambda_i=0 \text{ or } 3} \{\lambda_1\lambda_2\lambda_3\, 3\} + \cdots (\text{S-functions of more than 4 parts}).$$

Explicitly,

$$= \{0\} + \{3\} + (\{03\} + \{33\}) + (\{003\} + \{303\}$$

$$+ \{033\} + \{333\}) + (\{0003\} + \{3003\} + \{0303\} + \{3303\}$$

$$+\{0033\} + \{3033\} + \{0333\} + \{3333\} + \cdots (\text{S-functions of}$$

$$\text{more than 4 parts})$$

Upon modification lots of terms are null. The remaining standard terms are

$$\pi_i(1+x_i^{\,3})$$

$$= \{0\} + \{3\} + (-\{21\} + \{3^2\}) + (\{1^3\} - \{321\}$$

$$+ \{2^3\} + \{3^3\}) + \{0 + \{31^3\} + 0 - \{3^2 21\}$$

$$+ 0 + \{32^3\} + 0 + \{3^4\}) + \cdots (\text{S-functions of more than 4 parts})$$

By observation it is clear that each non-standard S-function of k parts with $k \leq 4$ is a 'joint' of the S-functions in the *elementary set* χ:

$$\chi = \{\{03\},\ \{003\},\ \{0^m 3\} \text{ with } m>2,\ \{033\},\ \{3^n\} \text{ with } n \geq 0\}$$

where $\{\lambda\}$ (of k parts) is a '*joint*' of $\{\rho\}$ (of ℓ parts with $\ell<k$) and $\{\sigma\}$ (of $k-\ell$ parts) if

$$\lambda_i = \rho_i \quad \text{for } i=1,\cdots,\ell$$

$$\lambda_{\ell+i} = \sigma_i \quad \text{for } i=1,\cdots,k-\ell$$

and denoted as $\{\lambda_1\cdots\lambda_k\} = \{\rho_1\cdots\rho_\ell:\sigma_1\cdots\sigma_{k-\ell}\} = \{\rho\}:\{\sigma\}$.

For example:

$$\{303\} = \{3:03\},\quad \{3003\} = \{3:003\}$$

$$\{0303\} = \{03:03\},\quad \{3303\} = \{33:03\}$$

$$\{0033\} = \{003:3\},\quad \{3033\} = \{3:033\}$$

$$\{0333\} = \{033{:}3\}, \text{ etc.}$$

By the method of mathematical induction, it is easy to show that the statement holds for arbitrary integer k, that is, any non-standard S-function (of k-parts) in the series $\pi_i(1+x_i^{\,3})$ is a unique 'joint' of the elements in the set χ.

To construct non-vanishing standard S-functions of the series, we first modify each term in set χ to yield the *standard elementary* set:

$$\chi_s = \{\ -\{21\},\ \{1^3\},\ \{2^3\},\ \{3^n\}\}.$$

The members of χ_s are called *elementary structures* since they are the elementary building blocks from which all the non-vanishing standard S-functions in the series can be constructed by 'joining'. By the modification rule $\{\lambda_1\cdots\lambda_i\ \ \lambda_{i+1}+1\cdots\} = 0$, $\{\lambda\}$ vanishes if $\lambda_{i+1} = \lambda_i + 1$, the 'joining' must be done according to some correct 'order of precedence' described by the *precedence diagram*. In this case we have

$$\overset{1}{\boxed{3^n}} \longrightarrow \overset{p(\geq 0)}{\boxed{2^3}} \longrightarrow \overset{0 \text{ or } 1}{\boxed{-21}} \longrightarrow \overset{q(\geq 0)}{\boxed{1^3}} \qquad (\text{with } n \geq 0)$$

which says, to construct the non-vanishing standard S-functions of the series $\pi_i(1+x_i^{\,3})$:

(i) take 1 copy of $\{3^n\}$ (with $n \geq 0$)

(ii) take $p(\geq 0)$ copies of $\{2^3\}$ and join it to the right hand side of (i)

(iii) take either 0 or 1 copy of $-\{21\}$ and join it to the right hand side of (ii)

(iv) take q (≥ 0) copies of $\{1^3\}$ and join it to the right hand side of (iii).

Thus the general standard terms are:

$$\{3^n\ 2^{3p}\ 1^{3q}\} \quad \text{and} \quad -\{3^n\ 2^{3p+1}\ 1^{3q+1}\}$$

where the negative sign comes from $-\{21\}$. Thus we have shown by actual construction that

$$\pi_i(1+x_i{}^3) = \sum_{n,p,q=0}^{\infty} (\{3^n2^{3p}1^{3q}\} - \{3^n2^{3p+1}1^{3q+1}\}).$$

Statement 5 : The standard S-function content for $\pi_i(1-x_i{}^3)$ series is

$$\pi_i(1-x_i{}^3) = \sum_{n,p,q=0}^{\infty} (-1)^{n+q}(\{3^n2^{3p}1^{3q}\}+\{3^n2^{3p+1}1^{3q+1}\})$$

Proof 5 : From table I.4.5 and the result above we see that

$$\pi_i(1-x_i{}^3) = \sum_{n,p,q=0}^{\infty} (-1)^{\omega_\lambda/3}(\{3^n2^{3p}1^{3q}\}-\{3^n2^{3p+1}1^{3q+1}\})$$

i.e. the S-function content for $\pi_i(1-x_i{}^3)$ differs from that for $\pi_i(1+x_i{}^3)$ only by the phase factor $(-1)^{\omega_\lambda/3}$

for $\{\lambda\} = \{3^n2^{3p}1^{3q}\}$, $\omega_\lambda = 3n+6p+3q$, $(-1)^{\omega_\lambda/3} = (-1)^{n+q}$

for $\{\lambda\} = \{3^n2^{3p+1}1^{3q+1}\}$, $\omega_\lambda = 3n+6p+3q+3$, $(-1)^{\omega_\lambda/3} = -(-1)^{n+q}$.

Thus the result follows.

Statement 6 : The standard S-function content for $\pi_i(1-x_i{}^3)^{-1}$ series is

$$\pi_i(1-x_i{}^3)^{-1} = \sum_{m;s\geq t=0}^{\infty} (\{m+3s\ m+3t\ m\} - \{m+3s+2\ m+3t+1\ m\})$$

Proof 6 : Since $\pi_i(1-x_i{}^3)^{-1}$ is the conjugate series of $\pi_i(1+x_i^3)$ it is only necessary to take the conjugate of S-function content of $\pi_i(1+x_i{}^3)$. Thus

$$\pi_i(1-x_i{}^3)^{-1} = \sum_{n,p,q=0}^{\infty} (\{3^n2^{3p}1^{3q}\} - \{3^n2^{3p+1}1^{3q+1}\}),$$

$$= \sum_{m;s\geq t=0}^{\infty} (\{m+3s\ m+3t\ m\} - \{m+3s+2\ m+3t+1\ m\})$$

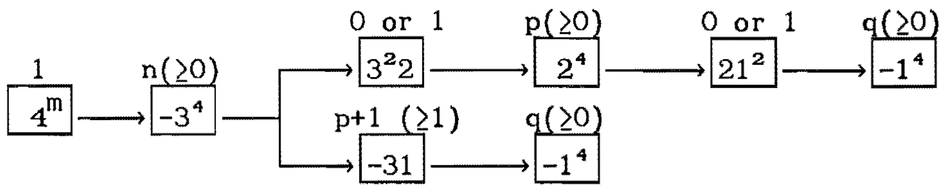
where $m=n$, $s=p+q \geq t=p$.

Results from table I.4.7:

Proofs for the results in this table are exactly the same as for table I.4.6. We merely point out that the standard elementary set for $\underset{i}{\pi}(1+x_i{}^4)$ is

$$\chi_s = \{-\{1^4\},\ \{21^2\},\ \{2^4\},\ -\{31\},\ \{3^22\},\ -\{3^4\},\{4^m\}\ m\geq 0\}$$

and the precedence diagram is



The non-vanishing standard S-functions in the series are the 'joint' of elementary structures according to the order of precedence, and the number of copies which can be taken are as given above each box.

The standard S-functions in $\pi_i(1+x_i{}^4)$ series involves 4,3,2,1 only as parts and the weight ω_λ are integral multiples of four. The content for $\underset{i}{\pi}(1-x_i{}^4)$ differ from this series by only the phase factor $(-1)^{\omega_\lambda/4}$, $\pi_i(1+x_i{}^4)^{-1}$ and $\pi_i(1-x_i{}^4)^{-1}$ are conjugate to $\pi_i(1+x_i{}^4)$ and $\pi_i(1-x_i{}^4)$ respectively and involve at most four parts.

In general, $\pi_i(1\pm x_i{}^k)$ and $\pi_i(1\pm x_i{}^k)^{-1}$ are multiplicity free series and have the following common features for their standard expansions:

(i) ω_λ/k is an integer, i.e. the weight of each term $\{\lambda\}$ in the series is an integral multiple of k.

(ii) $\lambda_1 \leq k$, i.e. the largest part for any S-function in the series $\pi_i(1\pm x_i^k)$ can not exceed k.

(iii) $\ell(\lambda) \leq k$, i.e. the maximum length (the number of parts) for any S-function in the series $\pi_i(1\pm x_i^k)^{-1}$ can not exceed k.

Results From table I.4.8:

Statement 7 : The standard S-function content for $\pi_i(1+x_i+x_i^2)$ series is

$$\pi_i(1+x_i+x_i^2) = \sum_{p,q=0}^{\infty} \phi(q)\ \{2^p1^q\}$$

where

$$\phi(q) = \sum_{x=0}^{[q/2]} (-1)^x \begin{bmatrix} q-x \\ x \end{bmatrix} = \begin{cases} 1 & \text{of}\quad q=0,1 \bmod 6 \\ 0 & \quad\quad 2,5 \bmod 6 \\ -1 & \quad\quad 3,4 \bmod 6 \end{cases}$$

$[q/2]$ is the integer part of $q/2$, $\begin{bmatrix} m \\ k \end{bmatrix}$ is the binominal coefficient.

Proof 7: From table I.4.5, for k=2

$$\pi_i(1+x_i+x_i^2) = \sum_\lambda \begin{Bmatrix} 0 & 0\cdots 0\cdots \\ 1 & 1\cdots 1\cdots \\ 2 & 2\cdots 2\cdots \end{Bmatrix}$$

$$= \sum_{p,q=0}^{\infty} \text{mult}(2^p1^q)\ \{2^p1^q\}$$

we need to find the multiplicity mult(2^p1^q).

The 'standard elementary set' for the series is

$$\chi_s = \{\{2^p\},\ \{1^q\},\ -\{1^2\}\}$$

and the 'precedence diagram' is

$$\boxed{2^p} \longrightarrow \boxed{1^r} \longleftrightarrow \boxed{-1^2} \qquad\qquad r\geq 0$$

where the double arrow between $\{1^r\}$ and $-\{1^2\}$ indicates the fact that they have the same order of precedence. Thus to construct the standard term $\{2^p1^q\}$ of the series, we may take 1 copy of $\{2^p\}$; x copies of $-\{1^2\}$ with $x=0,1,2,\cdots[q/2]$ and $q-2x$ copies of $\{1\}$ and join them together in accordance to order of precedence. Multiplicity occurs due to the face that $-\{1^2\}$ and $\{1\}$ have the same order of precedence and can be arranged in different orders. The number of ways to achieve 1^q is the number of arrangements which is given by

$$\sum_{x=0}^{[q/2]} \begin{bmatrix} q-x \\ x \end{bmatrix} = \sum_{x=0}^{[q/2]} \frac{(q-x)!}{x!(q-2x)!}$$

Each 1^2 carries with it a - sign. Thus the multiplicity is

$$\text{mult } (2^p1^q) = \sum_{x=0}^{[q/2]} (-1)^x \begin{bmatrix} q-x \\ x \end{bmatrix} = \phi(q)$$

Using the binomial coefficient identity

$$\begin{bmatrix} n \\ k \end{bmatrix} = \begin{bmatrix} n-1 \\ k \end{bmatrix} + \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}$$

it can be shown by induction on q that

$$\phi(q) = \sum_{x=0}^{[q/2]} (-1)^x \begin{bmatrix} q-x \\ x \end{bmatrix} = \begin{cases} 1 & \text{if } q=0,1 \bmod 6 \\ 0 & \phantom{\text{if }} 2,5 \bmod 6 \\ -1 & \phantom{\text{if }} 3,4 \bmod 6 \end{cases}$$

as stated.

We note that $\pi_i(1-x_i+x_i^2)$ is the adjoint series of $\pi_i(1+x_i+x_i^2)$

and their contents differ by the phase factor $(-1)^{\omega} = (-1)^{2p+q} = (-1)^q$ only, while $\pi_i(1-x_i+x_i^2)^{-1}$ and $\pi_i(1+x_i+x_i^2)^{-1}$ are conjugate to $\pi_i(1+x_i+x_i^2)$ and $\pi_i(1-x_i+x_i^2)$ series respectively, hence the other results in table I.4.8 can be established immediately.

Results from table I.4.9-11 can be proved in a similar fashion.

Results from table I.4.12:

Statement 8 : The non-standard S-function expansion for $L^{\dagger}A_+$ $(=QA^+=E^+)$ series is

$$\pi_i(1+x_i) \underset{i<j}{\pi}(1+x_ix_j) = \sum_{\lambda}\left\{\begin{matrix} 0 & 0 & 0\cdots & 0 & \cdots \\ 1 & 1 & 1\cdots & 1 & \cdots \\ & -2 & 4\cdots & (-1)^{s+1}(2s-2) & \cdots \\ & -3 & 5\cdots & (-1)^{s+1}(2s-1) & \cdots \end{matrix}\right\}$$

summed over all non-standard S-function $\{\lambda\}$ in which $\lambda_1 = 0$ or 1, $\lambda_s = 0, 1, (-1)^{s+1}(2s-2)$ or $(-1)^{s+1}(2s-1)$ for $s = 2,3,\cdots$.

Proof 10: Consider $\Delta(x_i)\, A_+$ first. Write

$$\Delta(x_i) \prod_{i<j=2}^{n} (1+x_ix_j) = \prod_{i<j=2}^{n} (x_i-x_j)(1+x_ix_j)$$

$$= \prod_{i=1}^{n} (1-x_i^2)^{n-1} \prod_{i<j} \left[\frac{x_i}{1-x_i^2} - \frac{x_j}{1-x_j^2}\right]$$

where the second factor is a generalized Vandermonde determinant, thus

$$\Delta(x_i) \prod_{i<j} (1+x_ix_j) = \prod_{i=1}^{n} (1-x_i^2)^{n-1} \left| \left[\frac{x_t}{1-x_t^2}\right]^{n-s} \right|$$

$$= \prod_{i=1}^{n} |x_t^{n-s}(1-x_t^2)^{s-1}| \quad \text{(by prop. 1)}$$

Apply the following elementary operations on colums c(s) for s=1,$\cdots$,n new columns c'(s) are obtained:

$$c(1) \rightarrow c'(1) = x_t^{n-1}$$

$$c(2) \rightarrow c'(2) = x_t^{n-2} - x_t^{n}$$

$$2c(1) + c(3) \rightarrow c'(3) = x_t^{n-3} + x_t^{n+1}$$

$$3c(2) + c(4) \rightarrow c'(4) = x_t^{n-4} - x_t^{n+2}$$

$$-6c(1) + 4c'(3) + c(5) \rightarrow c'(5) = x_t^{n-5} + x_t^{n+3}, \text{ etc.}$$

Thus

$$\Delta(x_i) \prod_{i<j=2}^{n} (1+x_i x_j) = |x_t^{n-1},\ x_t^{n-s} + (-1)^{s+1}\ x_t^{n-s+(2s-2)}|$$

$$\text{with } 2 \leq s \leq n \qquad \text{(by prop. 2)}$$

Multiply it by $\prod_{i=1}^{n} (1+x_i)\ \Delta(x_i)^{-1}$ to get

$$\prod_{i=1}^{n} (1+x_i) \prod_{i<j=2}^{n} (1+x_i x_j)$$

$$= \prod_{i=1}^{n} (1+x_i)\,|x_t^{n-1},\ x_t^{n-s}+(-1)^{s+1}x_t^{n-s+(2s-2)}|\,\Delta(x_i)^{-1}$$

$$= |x_t^{n-1}+x_t^{n},\ x_t^{n-s}+x_t^{n-s+1}+(-1)^{s+1}\ x_t^{n-s+(2s-2)} +(-1)^{s+1}\ x_t^{n-s+(2s-1)}|\,\Delta(x_i)^{-1}$$

$$= \sum_{\lambda} |x_t^{n-s+\lambda_s}| / |x_t^{n-s}| \qquad \text{(by prop.3)}$$

$$= \sum_{\lambda} \{\lambda_1\ \lambda_2 \cdots \lambda_n\}$$

where $\lambda_1 = 0$ or 1, $\lambda_s = 0,\ 1,\ (-1)^{s+1}(2s-2)$ or $(-1)^{s+1}(2s-1)$ for $s = 2,3,\cdots n$. Let $n \rightarrow \infty$, we obtain finally

$$\prod_{i}(1+x_i) \prod_{i<j} (1+x_i x_j) = \sum_{\lambda} \begin{Bmatrix} 0 & 0 & 0 & 0\cdots & 0 & \cdots \\ 1 & 1 & 1 & 1\cdots & 1 & \cdots \\ & -2 & 4 & -6\cdots & (-1)^{s+1}(2s-2) & \cdots \\ & -3 & 5 & -7\cdots & (-1)^{s+1}(2s-1) & \cdots \end{Bmatrix}$$

Note: the $(-1)^{s+1}$ in λ_s is understood to be a phase factor to be taken outside the curly bracket.[1] e.g. $\{0\text{-}2\} = -\{02\}$ $\{0\text{-}31\text{-}7\} = (-1)^2\{0317\}$, etc.

Statement 11 : The non-standard S-function for $L'A_+^{-1}$ $(=PB^+)$ series is

$$\pi_i(1+x_i)^{-1} \pi_{i<j}(1+x_ix_j)^{-1}$$

$$= \sum_\lambda (-1)^{\omega_\beta/2}\{(-1)^{m_1}(\beta_1+m_1)\ (-1)^{m_2}(\beta_2+m_2)\cdots$$

$$(-1)^{m_{2\ell}}(\beta_{2\ell}+m_{2\ell})\ (-1)^{m_{2\ell+1}}\ m_{2\ell+1}\cdots\}$$

summed over all non-standard S-functions $\{\lambda\}$ in which $\lambda_s=(-1)^{m_s}(\beta_s+m_s)$ for $s = 1,2,\cdots,2\ell$; $\lambda_s = (-1)^{m_s}\ m_s$ for $s>2\ell$. $m_s\geq 0$, $\ell\geq 0$, $\{\beta\}\in B$ with $\beta_1=\beta_2\geq\beta_3=\beta_4\geq\cdots\geq\beta_{2\ell-1}=\beta_{2\ell}>0$, $\ell(\beta)=2\ell$.

Proof : B^+ is known to be (from table I.4.1)

$$\pi_{i<j}(1+x_ix_j)^{-1} = \sum_\beta(-1)^{\omega_\beta/2}\{\beta\}$$

Express B^+ in determinantal form and multiply it by P to give

$$\pi_{i=1}^{n}(1+x_i)^{-1}\ \pi_{i<j=2}^{n}(1+x_ix_j)^{-1}$$

$$= \sum_{\beta\in B}(-1)^{\omega_\beta/2}\ \pi_{i=1}^{n(\geq 2\ell)}(1+x_i)^{-1}|x_t^{\,n-s+\beta_s}|\Delta(x_i)^{-1}$$

$$= \sum_{\beta\in B}(-1)^{\omega_\beta/2}|(1+x_t)^{-1}x_t^{\,n-s+\beta_s}|\Delta(x_i)^{-1} \qquad \text{(by prop.1)}$$

---

[1]We will adopt the convention in this section throughout that the minus sign appearing in the non-standard S-function expansion of a series is always taken to be a phase factor.

$$= \sum_{\beta\epsilon B} \sum_{m_s=0}^{\infty} (-1)^{\omega_\beta/2} |(-1)^{m_s} x^{n-s+\beta_s+m_s}| / |x_t^{n-s}|$$

Let $n \to \infty$,

$$\pi_{i}(1+x_i)^{-1} \pi_{i<j} (1+x_ix_j)^{-1}$$

$$= \sum_{\lambda}(-1)^{\omega_\beta/2}\{(-1)^{m_1}(\beta_1+m_1)(-1)^{m_2}(\beta_2+m_2)\cdots$$

$$(-1)^{m_{2\ell}}(\beta_{2\ell}+m_{2\ell})(-1)^{m_{2\ell+1}} m_{2\ell+1}\cdots\}$$

again, $(-1)^{m_s}$ for $s = 1,2,\cdots$ are phase factors to be taken outside the curly brackets, $\lambda_s=(-1)^{m_s}(\beta_s+m_s)$ for $s=1,2,\cdots,2\ell$, $\lambda_s=(-1)^{m_s}$ for $s>2\ell$ where 2ℓ is the length of β.

Other results in table I.4.12 can be similarly shown.

Results from table I.4.13:

Statement 12 : The standard S-function content for $E^+(=L^\dagger A_+ = QA^+)$ series is

$$\pi_{i}(1+x_i) \pi_{i<j} (1+x_ix_j) = \{0\} + \sum_{\substack{a_1>a_2>\cdots>a_r=0 \\ b_1>b_2>\cdots>b_r=0}}^{\infty} \sum_{r=1}^{\infty} \phi(r;\underline{a}\ \underline{b}) \begin{bmatrix} a_1a_2\cdots a_r \\ b_1b_2\cdots b_r \end{bmatrix}$$

where $\phi(r;\underline{a}\ \underline{b}) = \text{mult}\begin{bmatrix} a_1 & a_2\cdots a_r \\ b_1 & b_2\cdots b_r \end{bmatrix} = \text{mult}\begin{bmatrix} a_1 \\ b_1 \end{bmatrix}\text{mult}\begin{bmatrix} a_2 \\ b_2 \end{bmatrix}\cdots\text{mult}\begin{bmatrix} a_r \\ b_r \end{bmatrix}$

with

$$\text{mult}\begin{bmatrix} a_i \\ b_i \end{bmatrix} = \begin{cases} 0 & \text{if } a_i>b_i \quad \text{or} \quad a_i<b_{i+1} \\ 1 & \phantom{\text{if }} a_i=b_i \quad \text{or} \quad a_i=b_{i+1} \\ 2 & \phantom{\text{if }} a_i<b_i \quad \text{and} \quad a_i>b_{i+1} \end{cases}$$

for $i = 1,2,\cdots,r$.

Proof 12 : The non-standard S-function content for E^+ is

$$\pi_i(1+x_i)\ \pi_i(1+x_ix_j) = \left\{\begin{bmatrix} 0 & 0 & 0 & 0\cdots & & \cdots \\ 1 & 1 & 1 & 1\cdots & & \cdots \\ & -2 & 4 & -6 & (-1)^{s+1}(2s-2)\cdots \\ & -3 & 5 & -7 & (-1)^{s+1}(2s-1)\cdots \end{bmatrix}\right.$$

from table I.4.12. Write

$$\pi_i(1+x_i)\ \pi_{i<j}(1+x_ix_j) = E_0 + E_1 + E_2 + E_3 + \cdots$$

where E_r is the r-th rank S-function content of the series, $E_0 = \{0\}$.

The following terms are of Frobenius rank 1:

$$\{1^{n+1}\} = \begin{bmatrix} 0 \\ n \end{bmatrix}$$

$$\{0^{s-1}\ (-1)^{s+1}(2s-2)\ 1^n\} = \{s-1\ 1^{s-1+n}\} = \begin{bmatrix} s-2 \\ s-1+n \end{bmatrix}$$

$$\{0^{s-1}\ (-1)^{s+1}(2s-1)\ 1^n\} = \{s\ 1^{s-1+n}\} = \begin{bmatrix} s-1 \\ s-1+n \end{bmatrix}$$

with $s\geq 2$, $n\geq 0$. It is not hard to be convinced that these are the only non-vanishing rank one S-functions in the series. Thus

$$E_1 = \sum_{n,a=0}^{\infty} \left[\begin{bmatrix} 0 \\ n \end{bmatrix} + \begin{bmatrix} a \\ a+1+n \end{bmatrix} + \begin{bmatrix} a+1 \\ a+1+n \end{bmatrix}\right]$$

where $a=s-2\geq 0$. Write down the first few items explicitly for $a=0,1,2\cdots$ $n=0,1,2,\cdots$ and recombine them to yield

$$\begin{aligned} E_1 = & \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \end{bmatrix} + \begin{bmatrix} 0 \\ 3 \end{bmatrix} + \begin{bmatrix} 0 \\ 4 \end{bmatrix} + \cdots \\ & + \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \end{bmatrix} + \begin{bmatrix} 0 \\ 3 \end{bmatrix} + \begin{bmatrix} 0 \\ 4 \end{bmatrix} + \cdots \\ & + \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 \\ 3 \end{bmatrix} + \begin{bmatrix} 1 \\ 4 \end{bmatrix} + \cdots \end{aligned}$$

$$
\begin{array}{rrrrl}
+\begin{bmatrix}1\\2\end{bmatrix} & +\begin{bmatrix}1\\3\end{bmatrix} & +\begin{bmatrix}1\\4\end{bmatrix} & +\cdots \\
+\begin{bmatrix}2\\2\end{bmatrix} & +\begin{bmatrix}2\\3\end{bmatrix} & +\begin{bmatrix}2\\4\end{bmatrix} & +\cdots \\
 & +\begin{bmatrix}2\\3\end{bmatrix} & +\begin{bmatrix}2\\4\end{bmatrix} & +\cdots \\
 & +\begin{bmatrix}3\\3\end{bmatrix} & +\begin{bmatrix}3\\4\end{bmatrix} & +\cdots \\
 & & +\begin{bmatrix}3\\4\end{bmatrix} & +\cdots \\
 & & +\begin{bmatrix}4\\4\end{bmatrix} & +\cdots \\
 & & & +\cdots
\end{array}
$$

$$
= \sum_{n,a=0}^{\infty} \left[\begin{bmatrix}a\\a\end{bmatrix} +2 \begin{bmatrix}a\\a+1+n\end{bmatrix} \right]
$$

$$
= \sum_{a,b=0}^{\infty} \phi(1;a,b) \begin{bmatrix}a\\b\end{bmatrix}
$$

where

$$
\phi(1;a,b) = \begin{cases} 0 & \text{if } a_1 > b_1 \\ 1 & \quad a_1 = b_1 \qquad \text{(self conjugate)} \\ 2 & \quad a_1 < b_1 \end{cases}
$$

E_2 can be obtained from E_1 simply by adding at the t-th position ($t>\ell$) a new entry $(-1)^{t+1}(2t-2)$ or $(-1)^{t+1}(2t-1)$ followed by a string of 1's to the original first rank S-function of length ℓ. Upon modification, the effect of adding the new parts is equivalent to '*hook*' the original rank one S-function into the new rank one S-function:

$$
\{0^{t-1}\ (-1)^{t+1}(2t-2)\ 1^n\} = \{t-1\ 1^{t-1+n}\} = \begin{bmatrix}t-2\\t-1+n\end{bmatrix}
$$

or

$$
\{0^{t-1}\ (-1)^{t+1}(2t-1)\ 1^n) = \{t\ 1^{t-1+n}\} = \begin{bmatrix}t-1\\t-1+n\end{bmatrix}
$$

example:

$$
\{0\text{-}31\} = \{21^2\} = \begin{bmatrix}1\\2\end{bmatrix} \ \epsilon\ E_1,
$$

$$
\{000\text{-}711\} = \{41^5\} = \begin{bmatrix}3\\5\end{bmatrix} \ \epsilon\ E_1,
$$

thus

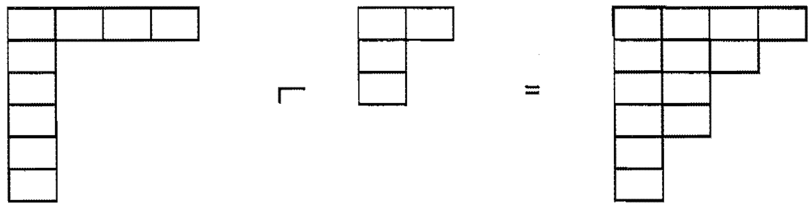
$$
\{0\text{-}31\ \text{-}711\} = \{432211\} = \begin{bmatrix}3 & 1\\5 & 2\end{bmatrix} = \begin{bmatrix}3\\5\end{bmatrix} \ulcorner \begin{bmatrix}1\\2\end{bmatrix} \ \epsilon\ E_2.
$$

where the symbol '$\ulcorner$' stands for 'hook'.

The corresponding Young diagram for

$$\begin{bmatrix}3\\5\end{bmatrix} \quad \ulcorner \quad \begin{bmatrix}1\\2\end{bmatrix} \quad = \quad \begin{bmatrix}3 & 1\\5 & 2\end{bmatrix}$$

is



In general, any second rank S-function in the series must take one of the following forms:

$$\{1^s\ 0\cdots0\ (-1)^{t+1}(2t-2)\ 1^n\}=\begin{bmatrix}t-2 & 0\\t-1+n & s-1\end{bmatrix}=\begin{bmatrix}t-2\\t-1+n\end{bmatrix}\ulcorner\begin{bmatrix}0\\s-1\end{bmatrix} \qquad (t\geq3,\ s\geq1)$$

$$\{1^s\ 0\cdots0\ (-1)^{t+1}(2t-1)\ 1^n\}=\begin{bmatrix}t-1 & 0\\t-1+n & s-1\end{bmatrix}=\begin{bmatrix}t-1\\t-1+n\end{bmatrix}\ulcorner\begin{bmatrix}0\\s-1\end{bmatrix} \qquad (t\geq2,\ s\geq1)$$

$$\{0^{s-1}\ (-1)^{s+1}(2s-2)\ 1^m\ 0\cdots0\ (-1)^{t+1}(2t-2)\ 1^n\}=\begin{bmatrix}t-2 & s-2\\t-1+n & s-1+m\end{bmatrix} \quad (t>s)$$
$$=\begin{bmatrix}t-2\\t-1+n\end{bmatrix}\ulcorner\begin{bmatrix}s-2\\s-1+m\end{bmatrix}$$

$$\{0^{s-1}\ (-1)^{s+1}(2s-2)\ 1^m\ 0\cdots0\ (-1)^{t+1}(2t-1)\ 1^n\}=\begin{bmatrix}t-1 & s-2\\t-1+n & s-1+m\end{bmatrix} \quad (t>s-1)$$
$$=\begin{bmatrix}t-1\\t-1+n\end{bmatrix}\ulcorner\begin{bmatrix}s-2\\s-1+m\end{bmatrix}$$

$$\{0^{s-1}\ (-1)^{s+1}(2s-1)\ 1^m\ 0\cdots0\ (-1)^{t+1}(2t-2)\ 1^n\}=\begin{bmatrix}t-2 & s-1\\t-1+n & s-1+m\end{bmatrix} \quad (t>s+1)$$
$$=\begin{bmatrix}t-2\\t-1+n\end{bmatrix}\ulcorner\begin{bmatrix}s-1\\s-1+m\end{bmatrix}$$

$$\{0^{s-1}\ (-1)^{s+1}(2s-1)\ 1^m\ 0\cdots0\ (-1)^{t+1}(2t-1)\ 1^n\}=\begin{bmatrix}t-1 & s-1\\t-1+n & s-1+m\end{bmatrix} \quad (t>s)$$
$$=\begin{bmatrix}t-1\\t-1+n\end{bmatrix}\ulcorner\begin{bmatrix}s-1\\s-1+m\end{bmatrix}$$

where $t>s+m+n_o$, $n_o\geq0$ is the number of 0 after 1^m, $0\leq m\leq t-1-s$.

By putting $t-2=a_1$, $s-2=a_2$ we obtain

$$E_2 = \sum_{n;a_1>a_2\geq0}\ \sum_{m=0}^{a_1-a_2-1}\left(\begin{bmatrix}a_1 & 0\\a_1+1+n & a_2+1\end{bmatrix}+\begin{bmatrix}a_1+1 & 0\\a_1+1+n & a_2+1\end{bmatrix}+\begin{bmatrix}a_1 & a_2\\a_1+1+n & a_2+1+m\end{bmatrix}\right)$$

$$+ \begin{bmatrix} a_1+1 & a_2 \\ a_1+1+n & a_2+1+m \end{bmatrix} + \begin{bmatrix} a_1 & a_2+1 \\ a_1+1+n & a_2+1+m \end{bmatrix} + \begin{bmatrix} a_1+1 & a_2+1 \\ a_1+1+n & a_2+1+m \end{bmatrix}\Bigg]$$

$$= \sum_{\substack{a_1 > a_2 = 0 \\ b_1 > b_2 = 0}}^{\infty} \phi\left[2;\underline{a},\underline{b}\right] \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix}.$$

In order to find multiplicity $\phi(2;\underline{a},\underline{b})$ we can write down the first few terms in the summation and then recombine them together, like we have done for E_1, or consider the following.

Since $\begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix}$ is the 'hook' of $\begin{bmatrix} a_2 \\ b_2 \end{bmatrix}$ with $\begin{bmatrix} a_1 \\ b_1 \end{bmatrix}$ we expect the multiplicity to be multiplicative, i.e.

$$\phi(2;\underline{a},\underline{b}) = \mathrm{mult}\begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix} = \mathrm{mult}\begin{bmatrix} a_1 \\ b_1 \end{bmatrix} \mathrm{mult}\begin{bmatrix} a_2 \\ b_2 \end{bmatrix}$$

where $\mathrm{mult}\begin{bmatrix} a_2 \\ b_2 \end{bmatrix} = \phi(1;a_2b_2)$, $\mathrm{mult}\begin{bmatrix} a_1 \\ b_1 \end{bmatrix}$ depends on $\phi(1;a_1,b_1)$ i.e. multiplicity of $\begin{bmatrix} a_1 \\ b_1 \end{bmatrix}$ if it were a free, non-hooked rank 1 S-function, and also on the way $\begin{bmatrix} a_1 \\ b_1 \end{bmatrix}$ hooks with $\begin{bmatrix} a_2 \\ b_2 \end{bmatrix}$. The only compatible solution is

$$\mathrm{mult}\begin{bmatrix} a_1 \\ b_1 \end{bmatrix} = \begin{cases} 0 & \text{if } a_1 > b_1 \text{ or } a_1 < b_2 \\ 1 & \phantom{\text{if }} a_1 = b_1 \text{ or } a_1 = b_2 \\ 2 & \phantom{\text{if }} a_1 < b_2 \text{ and } a_1 > b_2 \end{cases}$$

while for $\begin{bmatrix} a_2 \\ b_2 \end{bmatrix}$

$$\mathrm{mult}\begin{bmatrix} a_2 \\ b_2 \end{bmatrix} = \begin{cases} 0 & \text{if } a_2 > b_2 \\ 1 & \phantom{\text{if }} a_2 = b_2 \\ 2 & \phantom{\text{if }} a_2 < b_2 \end{cases}$$

The second set of constraints is dropped for $\begin{bmatrix} a_2 \\ b_2 \end{bmatrix}$ since there is nothing to be hooked into $\begin{bmatrix} a_2 \\ b_2 \end{bmatrix}$. Therefore

$$\phi(2;\underline{a},\underline{b}) = \mathrm{mult}\begin{bmatrix} a_1 \\ b_1 \end{bmatrix}\mathrm{mult}\begin{bmatrix} a_2 \\ b_2 \end{bmatrix} .$$

Consider now E_r for $r>2$. Write

$$E_r = \sum_{\substack{a_1 > a_2 > \cdots > a_r = 0 \\ b_1 > b_2 > \cdots > b_r = 0}}^{\infty} \phi(r;\underline{a},\underline{b}) \begin{bmatrix} a_1 \cdots a_r \\ b_1 \cdots b_r \end{bmatrix}$$

Since each term in E_r can be obtained from E_{r-1} where

$$E_{r-1} = \sum_{\substack{a_2 > a_3 > \cdots > a_r \\ b_2 > b_3 > \cdots > b_r}} \phi(r-1;\underline{a},\underline{b}) \begin{bmatrix} a_2 \cdots a_r \\ b_2 \cdots b_r \end{bmatrix}$$

by adding at the t-th position (t > length of $\begin{bmatrix} a_2 \quad a_r \\ b_2 \cdots b_r \end{bmatrix} = b_2+1$) a new entry $(-1)^{t+1}(2t-2)$ or $(-1)^{t+1}(2t-1)$ and a string of 1's afterwards to each term in E_{r-1}, $\begin{bmatrix} a_1 \cdots a_r \\ b_1 \cdots b_r \end{bmatrix}$ is a hook of $\begin{bmatrix} a_2 \cdots a_r \\ b_2 \cdots b_r \end{bmatrix} \epsilon E_{r-1}$ into $\begin{bmatrix} a_1 \\ b_1 \end{bmatrix}$ i.e.

$$\begin{aligned}
\begin{bmatrix} a_1 \cdots a_r \\ b_1 \cdots b_r \end{bmatrix} &= \begin{bmatrix} a_1 \\ b_1 \end{bmatrix} \ulcorner \begin{bmatrix} a_2 \cdots a_r \\ b_2 \cdots b_r \end{bmatrix}. \\
&= \begin{bmatrix} a_1 \\ b_1 \end{bmatrix} \ulcorner \begin{bmatrix} a_2 \\ b_2 \end{bmatrix} \ulcorner \begin{bmatrix} a_3 \cdots a_r \\ b_3 \cdots b_r \end{bmatrix} \\
&\vdots \\
&= \begin{bmatrix} a_1 \\ a_1 \end{bmatrix} \ulcorner \begin{bmatrix} a_2 \\ a_2 \end{bmatrix} \ulcorner \cdots \ulcorner \begin{bmatrix} a_i \\ a_i \end{bmatrix} \ulcorner \begin{bmatrix} a_{i+1} \\ b_{i+1} \end{bmatrix} \ulcorner \cdots \ulcorner \begin{bmatrix} a_r \\ a_r \end{bmatrix}.
\end{aligned}$$

by induction on $\ulcorner$. Hence $\phi(r;\underline{a},\underline{b}) = \text{mult}\begin{bmatrix} a_1 \cdots a_r \\ b_1 \cdots b_r \end{bmatrix} = \text{mult}\begin{bmatrix} a_1 \\ b_1 \end{bmatrix}$ $\text{mult}\begin{bmatrix} a_2 \\ b_2 \end{bmatrix} \cdots \text{mult}\begin{bmatrix} a_r \\ b_r \end{bmatrix}$ where $\text{mult}\begin{bmatrix} a_i \\ b_i \end{bmatrix}$ depends on multiplicity of $\begin{bmatrix} a_i \\ b_i \end{bmatrix}$ if it were an independent rank one S-function and also the way $\begin{bmatrix} a_{i+1} \\ b_{i+1} \end{bmatrix}$ hooks into it. Thus

$$\text{mult}\begin{bmatrix} a_i \\ b_i \end{bmatrix} = \begin{cases} 0 & \text{if} \quad a_i > b_i \quad \text{or } a_i < b_{i+1} \\ 1 & \qquad a_i = b_i \quad \text{or } a_i = b_{i+1} \\ 2 & \qquad a_i < b_i \quad \text{and } a_i > b_{i+1} \end{cases}$$

for $i = 1, 2, \cdots, r-1$

$$\mathrm{mult}\begin{bmatrix} a_r \\ b_r \end{bmatrix} = \begin{cases} 0 & \text{if} \quad a_r > b_r \\ 1 & \qquad a_r = b_r \\ 2 & \qquad a_r < b_r \end{cases}$$

The two sets of expressions can be combined into one which holds for $i=1,2\cdots,r$ if we take $a_{r+1}=b_{r+1}=0$ in E_r. Thus statement 12 is proved.

Here is a simple algorithm to evaluate the multiplicity of an arbitrary rank r S-function $\{\lambda\}$ in the E^+ series:

(i) Write the S-function in Frobenius notation $\{\lambda\} = \begin{bmatrix} a_1 \cdots a_r \\ b_1 \cdots b_r \end{bmatrix}$.

(ii) Check that all of the following relationships between adjacent pairs are satisfied, namely,

$$\begin{tikzcd} a_i \arrow[r, ">"] \arrow[d, "\leq"'] \arrow[dr, "\geq"] & a_{i+1} \arrow[d, "\leq"] \\ b_i \arrow[r] & b_{i+1} \end{tikzcd} \qquad \text{for } i=1,\cdots,r-1$$

if not, then the multiplicity is zero. Otherwise,

(iii) count the total number of '=' signs and denote it as e.

(iv) $\mathrm{mult}\begin{bmatrix} a_1 \cdots a_r \\ b_1 \cdots b_r \end{bmatrix} = 2^{r-e}$.

Examples:

$$\mathrm{mult}\begin{bmatrix} 3 & 2 & 0 \\ 4 & 3 & 1 \end{bmatrix} = 2^{3-1} = 4$$

$$\mathrm{mult}\begin{bmatrix} 8 & 6 & 5 & 4 & 1 & 0 \\ 8 & 7 & 6 & 4 & 2 & 1 \end{bmatrix} = 2^{6-4} = 4$$

$$\mathrm{mult}\begin{bmatrix} 5 & 4 & 1 \\ 6 & 3 & 0 \end{bmatrix} = 0$$

which agree with the results obtained by direct calculations using statement 12:

$$\begin{aligned} \mathrm{mult}\begin{bmatrix} 3 & 2 & 0 \\ 4 & 3 & 1 \end{bmatrix} &= \mathrm{mult}\begin{bmatrix} 3 \\ 4 \end{bmatrix} \ \mathrm{mult}\begin{bmatrix} 2 \\ 3 \end{bmatrix} \ \mathrm{mult}\begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ &= \quad 1 \quad \cdot \quad 2 \quad \cdot \quad 2 \end{aligned}$$

$$= \quad 4$$

$$\text{mult}\begin{bmatrix}8 & 6 & 5 & 4 & 1 & 0\\ 8 & 7 & 6 & 4 & 2 & 1\end{bmatrix} = \text{mult}\begin{bmatrix}8\\8\end{bmatrix}\ \text{mult}\begin{bmatrix}6\\7\end{bmatrix}\ \text{mult}\begin{bmatrix}5\\6\end{bmatrix}\ \text{mult}\begin{bmatrix}4\\4\end{bmatrix}\ \text{mult}\begin{bmatrix}1\\2\end{bmatrix}\ \text{mult}\begin{bmatrix}0\\1\end{bmatrix}$$

$$= 1 \cdot 1 \cdot 2 \cdot 1 \cdot 1 \cdot 2$$

$$= 4$$

$$\text{mult}\begin{bmatrix}5 & 4 & 1\\ 6 & 3 & 0\end{bmatrix} = \text{mult}\begin{bmatrix}5\\6\end{bmatrix}\ \text{mult}\begin{bmatrix}4\\3\end{bmatrix}\ \text{mult}\begin{bmatrix}1\\0\end{bmatrix}$$

$$= 2 \cdot 0 \cdot 0$$

$$= 0.$$

Standard S-function contents for L^m-family series displayed in table I.4.14 are derived in a rather different fashion, by making use of the following identities (Macdonald 1979, Littlewood 1950):

$$\prod_{i,\alpha} (1 + x_i y_\alpha) = \sum_\lambda \{\lambda\}(x_i)\cdot\{\lambda'\}(y_\alpha) \qquad \text{(I.4.2-3a)}$$

$$\prod_{i,\alpha} (1 - x_i y_\alpha)^{-1} = \sum_\lambda \{\lambda\}(x_i)\cdot\{\lambda\}(y_\alpha) \qquad \text{(I.4.2-3b)}$$

$$\prod_{i,\alpha} (1 - x_i y_\alpha) = \sum_\lambda (-1)^{\omega_\lambda}\{\lambda\}(x_i)\{\lambda'\}(y_\alpha) \qquad \text{(I.4.2-3c)}$$

$$\prod_{i,\alpha} (1 + x_i y_\alpha)^{-1} = \sum_\lambda (-1)^{\omega_\lambda}\{\lambda\}(x_i)\cdot\{\lambda\}(y_\alpha) \qquad \text{(I.4.2-3d)}$$

and choosing the special variables y_α to be

$$y_\alpha = \begin{cases} 1 & \text{for } \alpha = 1,2,\cdots,m \\ 0 & \text{for } \alpha > m \end{cases} \qquad \text{(I.4.2-4)}$$

yielding

$$Q^m : \prod_i (1 + x_i)^m = \sum_\lambda \phi_m^{\lambda'}\{\lambda\}$$

$$M^m : \prod_i (1 - x_i)^{-m} = \sum_\lambda \phi_m^{\lambda}\{\lambda\}$$

$$L^m : \prod_i (1 - x_i)^m = \sum_\lambda (-1)^{\omega_\lambda} \phi_m^{\lambda'}\{\lambda\}$$

and

$$P^m : \prod_i (1 + x_i)^m = \sum_\lambda (-1)^{\omega_\lambda} \phi_m^{\lambda}\{\lambda\}$$

where ϕ_m^λ is the S-function $\{\lambda\}$ in special variables y_α defined in (I.4.2-4). Explicitly, as given by Littlewood

$\phi_m^\lambda = f^{(\lambda)}/n!$ times the first λ_i terms for the array

$$\begin{array}{llll} m & m+1 & m+2 & m+3\cdots \\ m-1 & m & m+1 & m+2\cdots \\ m-2 & m-1 & m & m+1\cdots \\ \cdot & \cdot & \cdot & \cdot\ \cdots, \end{array}$$

$f^{(\lambda)}$ with $\lambda\vdash n$ is the degree (dimension) of the corresponding character $\{\lambda\}$ of the symmetric group S_n.

It is also interesting to note that by setting $y_\alpha = x_i$, we obtain the following S-function series identities

$$A^+C^+ = \prod_{i,j}(1 + x_ix_j) = \sum_\lambda\{\lambda\}\cdot\{\lambda'\}$$

$$BD = \prod_{i,j}(1 - x_ix_j)^{-1} = \sum_\lambda\{\lambda\}\cdot\{\lambda\}$$

$$AC = \prod_{i,j}(1 - x_ix_j) = \sum_\lambda(-1)^{\omega_\lambda}\{\lambda\}\cdot\{\lambda'\}$$

$$B^+D^+ = \prod_{i,j}(1 + x_ix_j)^{-1} = \sum_\lambda(-1)^{\omega_\lambda}\{\lambda\}\cdot\{\lambda\}$$

or

$$\sum_\lambda\{\lambda\}\cdot\{\lambda'\} = \sum_{\alpha,\gamma}\{\alpha\}\cdot\{\gamma\} \qquad \text{(I.4.2-5)}$$

$$\sum_\lambda\{\lambda\}\cdot\{\lambda\} = \sum_{\beta,\sigma}\{\beta\}\cdot\{\delta\} \qquad \text{(I.4.2-6)}$$

where α, β, σ, and γ are S-function contents of A, B, C and D series as defined in table I.3.1. $\{\lambda'\}$ is the conjugate of $\{\lambda\}$.

More than forty new S-function series have been obtained in this chapter. We note that D_+ series and the identity for AC arises in the study of the non-compact group Sp(2N,R) and U(p,q) respectively. (King and Wybourne 1985, Yang and Wybourne 1986). The L^m-family series can be used to obtain the chain branching rule, e.g.

$$U(n) \downarrow U(n-1) \downarrow \cdots \downarrow U(n-m) \qquad m<n$$

$$\{\lambda\} \downarrow \{\lambda/M\} \downarrow \cdots \downarrow \{\lambda/M^m\}. \qquad (I.4.2-7)$$

APPENDIX I

The aim of this appendix is to prove the relationship between the generating function for an S-function series of type I, II or III and its inverse, conjugate and adjoint series, as given in table I.3.3.

Proposition 1: For two arbitrary S-function series S and T with generating function S(x) and T(x), S-function expansion $S = \sum_{\lambda}\{\lambda\}$, and $T = \sum_{\mu}\{\mu\}$, the inverse, conjugate and adjoint of the product series ST is given by

(a) $(ST)^{-1} = S^{-1}\, T^{-1}$

(b) $(ST)' = S'\, T'$

(c) $(ST)^{\dagger} = S^{\dagger}\, T^{\dagger}$

Proof 1: (a) This is obviously true from the definition of inverse series.

(b) Multiplying series S and T using the Littlewood-Richardson rule we obtain

$$ST = \sum_{\lambda}\{\lambda\} \sum_{\mu}\{\mu\}$$
$$= \sum_{\lambda,\sigma,\mu} \Gamma^{\sigma}_{\lambda\mu}\, \{\sigma\}$$

The conjugate of the product series is

$$
\begin{aligned}
(ST)' &= \sum_{\sigma} \Gamma^{\sigma}_{\lambda\mu} \{\sigma\}' \\
&= \sum_{\sigma'} \Gamma^{\sigma'}_{\lambda'\mu'} \{\sigma\}' \\
&= \sum_{\lambda'} \{\lambda\}' \sum_{\mu'} \{\mu\}' \\
&= S'\ T'
\end{aligned}
$$

where the property of the Littlewood-Richardson's coefficient

$$\Gamma^{\sigma}_{\lambda\mu} = \Gamma^{\sigma'}_{\lambda'\mu'}$$

has been used.

(c) Use the result of (a) and (b)

$$
\begin{aligned}
(ST)^{\dagger} &= ((ST)^{-1})' \\
&= (S^{-1}\ T^{-1})' \\
&= S^{\dagger}\ T^{\dagger}.
\end{aligned}
$$

Proposition 2: Let S_1 be a type I series generated by $S_1(x_i) = \pi_i(1 - f_1(x_i))$ where $f_1(x_i) = \sum_{k=1}^{n} c_k\ x_i^k$, then the conjugate and adjoint series of $S_1(x_i)$ is

$$
\begin{aligned}
&(a)\quad S_1'(x_i) = S_1(-x_i)^{-1} \\
&(b)\quad S_1^{\dagger}(x_i) = S_1(-x_i)
\end{aligned}
$$

Proof: (a) Since $1 - f_1(x_i)$ is a polynomial of degree n, it has n roots (some may be degenerate) $\alpha_1^{-1}, \alpha_2^{-1}, \cdots, \alpha_n^{-1}$ over the complex field. We can write

$$1 - f_1(x_i) = \prod_{s=1}^{n} (1 - \alpha_s x_i)$$

and
$$S_1(x_i) = \pi_i(1 - f_1(x_i))$$
$$= \pi_{s=1}^{n} \pi_i(1 - \alpha_s x_i)$$
$$= \pi_{s=1}^{n} L(\alpha_s x_i)$$

Under conjugation,

$$L(y_i)' = P(y_i) = L(-y_i)^{-1}$$

Thus the conjugate of $S_1(x_i)$ is

$$S_1'(x_i) = S_1(x_i)' = \pi_{s=1}^{n} L(\alpha_s x_i)' \qquad \text{by proposition 1(b)}$$
$$= \pi_{s=1}^{n} L(-\alpha_s x_i)^{-1}$$
$$= \pi_i \pi_{s=1}^{n} (1 - \alpha_s(-x_i))^{-1}$$
$$= \pi_i (1 - f(-x_i))^{-1}$$
$$= S_1 (-x_i)^{-1}$$

(b) $$S_1^{\dagger}(x_i) = (S_1(x_i)')^{-1}$$
$$= S_1(-x_i).$$

Proposition 3: Let S_2 be an S-function series of type II, generated by $\pi_{i<j}(1-f_2(x_i x_j))$ where $f_2(x_i x_j) = \sum_{k=1}^{n} c_k(x_i x_j)^k$, then the conjugate and adjoint of S_2 is

(a) $$S_2'(x_i x_j)_{i<j} = S_2(x_i x_j)_{i\leq j}$$

(b) $$S_2^{\dagger}(x_i x_j)_{i<j} = S_2(x_i x_j)_{i\leq j}^{-1}$$

Proof 3: (a) The proof is very similar to 2(a). Factorize $1 - f_2(x_i x_j)$ formally as

$$1 - f_2(x_1 x_2) = \prod_{s=1}^{n} (1 - \alpha_s x_i x_j)$$

$$\begin{aligned} \underset{i<j}{S_2(x_i x_j)} &= \prod_{s=1}^{n} \prod_{i<j} (1 - \alpha_s x_i x_j) \\ &= \prod_{s=1}^{n} A(\alpha_s x_i x_j). \end{aligned}$$

Under conjugation, A series goes to C series, i.e.

$$\underset{i<j}{A'(y_i y_j)} = C(y_i y_j) = \underset{i\leq j}{A(y_i y_j)}$$

Thus

$$\begin{aligned} S_2'(x_i x_j) = \underset{i<j}{S_2(x_i x_j)'} &= \prod_{s=1}^{n} A(\alpha_s x_i x_j)' \\ &= \prod_{s=1}^{n} C(\alpha_s x_i x_j) \\ &= \prod_{i\leq j} \prod_{s=1}^{n} (1 - \alpha_s x_i x_j) \\ &= \prod_{i\leq j} (1 - f_2(x_i x_j)) \\ &= S_2(x_i x_j) \quad \text{with} \quad i\leq j. \end{aligned}$$

(b) $$\begin{aligned} S_2^{\dagger}(x_i x_j) &= (S_2(x_i x_j)')^{-1} \\ &= S_2(x_i x_j)^{-1} \quad \text{with} \quad i\leq j. \end{aligned}$$

APPENDIX II

We prove in this appendix the identities of plethysms of S-function series which were used in section I.4.1.

Identity 1: $(\sum_{\lambda}\{\lambda\})\otimes[L]^{-1} = -(\sum_{\lambda}\{\lambda\})\otimes[L]$

Identity 2: $\{\lambda\}\otimes[L]\cdot\{\mu\}\otimes[L] = (\{\lambda\}+\{\mu\})\otimes[L]$

Identity 3: $(\{\lambda\}\otimes[L])' = \begin{cases}\{\lambda\}'\otimes[L] & \text{if } \omega_\lambda \text{ even}\\ \{\lambda\}'\otimes[L]' & \text{if } \omega_\lambda \text{ odd}\end{cases}$

Identity 4: $((\sum_{\lambda}\{\lambda\})\otimes[L])^{-1} = (\sum_{\lambda}\{\lambda\})\otimes[L]^{-1}$

Identity 5: $[L](-x_i^{\,k}) = -p_k\otimes[L]' = p_k\otimes[L]^{\dagger}$

The following properties of S-function plethysms are used in the proofs (Wybourne 1970):

$$\{\lambda\}\otimes(\{\mu\}+\{\sigma\}) = \{\lambda\}\otimes\{\mu\} + \{\lambda\}\otimes\{\sigma\} \tag{A2-1}$$

$$(\{\lambda\} + \{\mu\})\{\sigma\} = \sum_{\tau} \{\lambda\}\otimes\{\sigma/\tau\}\cdot\{\mu\}\otimes\{\tau\} \tag{A2-2}$$

$$(\{\lambda\} - \{\mu\})\otimes\{\sigma\} = \sum_{\tau}(-1)^{\omega_\tau}\{\lambda\}\otimes\{\sigma/\tau\}\cdot\{\mu\}\otimes\{\tau'\} \tag{A2-3}$$

$$(\{\lambda\}\otimes\{\mu\})' = \begin{cases}\{\lambda'\}\otimes\{\mu\} & \text{if } \omega_\lambda \text{ even} \quad \text{(A2-4a)}\\ \{\lambda'\}\otimes\{\mu'\} & \text{if } \omega_\lambda \text{ odd} \quad \text{(A2-4b)}\end{cases}$$

$$\{0\}\otimes\{\lambda\} = \begin{cases}\{0\} & \text{if } \ell(\lambda) = 0,1 \quad \text{(A2-5a)}\\ 0 & \text{if } \ell(\lambda) > 1 \quad \text{(A2-5b)}\end{cases}$$

Proof 1: First, we show that

$$-\{\lambda\}\otimes\{m\} = \{\lambda\}\otimes(-1)^m\{1^m\} \tag{A2-6a}$$

and

$$-\{\lambda\}\otimes(-1)^m\{m\} = \{\lambda\}\otimes\{1^m\} \tag{A2-6b}$$

The left hand side of (A2-6a) is

$$\begin{aligned}&-\{\lambda\}\otimes\{m\}\\ &= (\{0\} - (\{\lambda\} + \{0\}))\otimes\{m\}\\ &= \sum_{n=0}^{m} (-1)^n\{0\}\otimes\{m-n\}\cdot(\{\lambda\}+\{0\})\otimes\{1^n\}\end{aligned}$$

$$= \sum_{n=0}^{m} \sum_{p=0}^{n} (-1)^n(\{\lambda\}\otimes\{1^{n-p}\}\cdot\{0\}\otimes\{1^p\})$$

$$= \sum_{n=0}^{m} (-1)^n(\{\lambda\}\otimes\{1^n\}+\{\lambda\}\otimes\{1^{n-1}\})$$

$$= \{\lambda\}\otimes(\{0\}-\{1\}+\{1^2\}- \cdots +(-1)^m\{1^m\})$$

$$+ \{\lambda\}\otimes(-\{0\}+\{1\}-\{1^2\}+ \cdots +(-1)^m\{1^{m-1}\})$$

$$= \{\lambda\}\otimes(-1)^m\{1^m\}$$

i.e. $$- \{\lambda\}\otimes\{m\} = \{\lambda\}\otimes(-1)^m\{1^m\}$$

Multiply both sides of (A2-6a) by $(-1)^m$ and (A2-6b) is obtained.

To establish identity I, use the result (A2-6a,b) and sum over the value of m to yield

$$\sum_{m=0}^{\infty} - \{\lambda\}\otimes\{m\} = \sum_{m=0}^{\infty} \{\lambda\}\otimes(-1)^m\{1^m\}$$

$$- \{\lambda\}\otimes \sum_{m=0}^{\infty} \{m\} = \{\lambda\}\otimes\sum_{m}(-1)^m\{1^m\}$$

i.e. $$\{\lambda\}\otimes L^{-1} = -\{\lambda\}\otimes L$$

$$\{\lambda\}\otimes M^{-1} = -\{\lambda\}\otimes M$$

also $$\sum_{m=0}^{\infty} -\{\lambda\}\otimes(-1)^m\{m\} = \sum_{m=0}^{\infty} \{\lambda\}\otimes\{1^m\}$$

$$-\{\lambda\}\otimes \sum_{m=0}^{\infty} (-1)^m\{m\} = \{\lambda\}\otimes \sum_{m=0}^{\infty} \{1^m\}$$

i.e. $$\{\lambda\}\otimes P^{-1} = -\{\lambda\}\otimes P$$

$$\{\lambda\}\otimes Q^{-1} = -\{\lambda\}\otimes Q$$

Thus $$\{\lambda\}\otimes[L]^{-1} = -\{\lambda\}\otimes[L].$$

The above proof will not be altered if $\{\lambda\}$ is replaced by $\sum_{\lambda}\{\lambda\}$. Hence the proof for identity 1 is completed.

Proof 2: Consider the case $[L] = Q$.

$$
\begin{aligned}
&(\{\lambda\} + \mu\})\otimes Q \\
&= (\{\lambda\} + \{\mu\})\otimes \sum_{m=0}^{\infty} \{1^m\} \\
&= \sum_{m=0}^{\infty} (\{\lambda\} + \{\mu\})\otimes\{1^m\} \\
&= \sum_{m=0}^{\infty} \sum_{n=0}^{m} \{\lambda\}\otimes\{1^{m-n}\}\cdot\{\mu\}\otimes\{1^n\} \\
&= \sum_{m=0}^{\infty} \{\lambda\}\otimes\{1^m\}\cdot\{\mu\}\otimes(\{0\} + \{1\} + \{1^2\} + \{1^3\}+\cdots) \\
&= \sum_{m=0}^{\infty} \{\lambda\}\otimes\{1^m\}\cdot \sum_{n=0}^{\infty} \{\mu\}\otimes\{1^n\} \\
&= \{\lambda\}\otimes Q\cdot\{\mu\}\otimes Q.
\end{aligned}
$$

where we have used the property of the infinite series

$$\sum_{m=0}^{\infty} \{1^{m-n}\} = \sum_{m=0}^{\infty} \{1^m\}$$

and assumed $\qquad \{\lambda\}\otimes\{1^{-s}\} = 0 \quad (s>0)$

Proofs for the case [L] = L, M, and P can be obtained similarly.

Proof 3: Identity 3 holds for arbitrary S-function series. Let $S = \sum_{\sigma} z_{\sigma}\{\sigma\}$.

$$
\begin{aligned}
&(\{\lambda\}\otimes S)' \\
&= \sum_{\sigma} z_{\sigma}(\{\lambda\}\otimes\{\sigma\})' \\
&= \begin{cases} \sum_{\sigma} z_{\sigma}\{\lambda'\}\otimes\{\sigma\} & \text{if } \omega_{\lambda} \text{ even} \\ \sum_{\sigma} z_{\sigma}\{\lambda'\}\otimes\{\sigma'\} & \text{if } \omega_{\lambda} \text{ odd} \end{cases} \\
&= \begin{cases} \{\lambda'\}\otimes S & \text{if } \omega_{\lambda} \text{ even} \\ \{\lambda'\}\otimes S' & \text{if } \omega_{\lambda} \text{ odd} \end{cases}
\end{aligned}
$$

Proof 4: By definition, T is an inverse series of S if TS = ST = {0} = 1.

$$
\begin{aligned}
&(\{\lambda\}\otimes[L])\cdot(\{\lambda\}\otimes[L]^{-1}) \\
&= (\{\lambda\}\otimes[L])\cdot(-\{\lambda\}\otimes[L]) && \text{(by identity 1)} \\
&= (\{\lambda\} - \{\lambda\})\otimes[L] && \text{(by (identity 2)} \\
&= \{0\}.
\end{aligned}
$$

Hence

$$(\{\lambda\}\otimes[L])^{-1} = \{\lambda\}\otimes[L]^{-1}$$

Replace $\{\lambda\}$ by $\sum_{\lambda}\{\lambda\}$, identity 4 is obtained.

Proof 5: From the properties of L-family series given in table I.3.1 we have

$$
\begin{aligned}
[L](-x_i) &= [L]^{\dagger}(x_i) \\
&= \{1\}\otimes[L]^{\dagger} \\
&= -\{1\}\otimes[L]' && \text{by (identity 1)}
\end{aligned}
$$

Replace the variable x_i formally by x_i^k and use the general results $S(x_i^k) = p_k\otimes S$ where p_k is the power sum symmetric function, we obtain

$$
\begin{aligned}
[L](-x_i^k) &= [L]^{\dagger}(x_i^k) \\
&= p_k\otimes[L]^{\dagger} \\
&= -p_k\otimes[L]'.
\end{aligned}
$$

CHAPTER II

EXTENDED POINCARÉ SUPERSYMMETRY, ROTATION GROUPS AND BRANCHING RULES

II.1 INTRODUCTION

The even-dimensional rotation group SO_{2k} continues to find wide applications in physical problems. Important examples arise in the interacting boson model of nuclei (Arima and Iachello 1976), extended Poincaré supersymmetry (Strathdee 1987) and in superstring theories (Green and Schwarz 1984, Ramond 1985). The properties of the basic spin irreps of SO_n are of special significance. The analysis of the antisymmetric powers of a spin irrep is an important problem in supergravity theories (Curtright 1982, Bergshoeff and de Roo 1982, 1984). A complete resolution of the second and third powers of the basic spin irrep of SO_n together with a prescription for analysing the fourth power of these irreps has been given (King et al. 1981). A complete reduction of all antisymmetric powers of the basic spin irrep of SO_{10} has been derived (Black and Wybourne 1983).

The study of the properties of the even-dimensional rotation group SO_{2k} is complicated by the occurrence of pairs of conjugate irreps: $\Delta_{\pm}$, $[\Delta;\lambda]_{\pm}$, $[\lambda]_{\pm}$ (with $\lambda_k \neq 0$) which must be carefully distinguished at all stages. A compact description of the resolution of the Kronecker products of irreps of SO_{2k} has been given (Black et al. 1983). Branching rules for irreps of SO_{2k} have in many cases been developed (King 1975; Black et al. 1983; Black and Wybourne 1983). A method for $E_8 \downarrow SO_{16}$ branching rules has also been given (Wybourne 1984). Properties of S-functions, especially S-function series played an essential part in these studies.

In this chapter, the decomposition of the basic spin irreps of the special orthogonal group SO_{2k} into irreps of $SO_{D-2} \times K$ which arises in extended Poincaré supersymmetry is determined. Several new branching rules for subgroups of SO_{2k} are developed. The arrangement of the chapter is as follows:

In section II.2, we examine the isomorphisms and automorphism of SO_n (table II.2.1) and give a complete resolution of all antisymmetric powers of the basic spin irreps of SO_n for $n \leq 11$ (table II.2.2). It then becomes possible to express any symmetrized powers of the basic spin irreps for $n \leq 10$ as sums of products of the antisymmetric powers. The symmetrized powers of the basic spin irreps, up to fourth power, are tabulated (table II.2.3).

In section II.3, a number of additional branching rules for important subgroups of SO_{2k} were introduced. The results draw heavily upon the properties of S-function series (King et al. 1981; Black et al. 1983).

In section II.4, we examine the decomposition of the basic spin irreps of SO_{2k} under the group-subgroup restriction $SO_{2k} \downarrow SO_{D-2} \times K$ where D is the spacetime dimension of the extended D-dimensional Poincaré supersymmetry and K is the automorphism group appropriate to D (Table II.4.1). This problem is of special significance in determining acceptable light-like representations. Group isomorphisms and automorphisms are exploited to give for $D \leq 10$ results that can readily be extended to arbitrary number N of supercharges. A general method for cases in which $D > 10$ is sketched. Useful tables of decompositions (Table II.4.3, II.4.5) are given.

Notation used here for irreps of classical groups are the *standard partition label* (Black et al. 1983). Different brackets { }, [ ], < >, are used for irreps of unitary, orthogonal and symplectic groups respectively. Appropriate modification rules (King 1970, Black et al

1983) are used to convert non-standard labels to standard forms in which the length of partition $\ell(\lambda)$ is restricted by the rank of the algebra associated with the group. Spinor and basic spin irreps of orthogonal groups are signified by $[\Delta;\lambda]$ and Δ respectively (King 1975).

For special orthogonal groups in even dimension, SO_{2k}, it is often convenient and sometimes necessary to introduce the 'sum' and 'difference' characters (Murnaghan 1938, Littlewood 1950, Butler and Wybourne 1969, King et al. 1981)

$$\begin{aligned}
[\lambda] &= [\lambda]_+ + [\lambda]_- && \lambda_k \neq 0 \\
[\lambda]'' &= [\lambda]_+ - [\lambda]_- && \lambda_k \neq 0 \\
\Delta &= \Delta_+ + \Delta_- \\
\Delta'' &= \Delta_+ - \Delta_- \\
[\Delta;\lambda] &= [\Delta;\lambda]_+ + [\Delta;\lambda]_- \\
[\Delta;\lambda]'' &= [\Delta;\lambda]_+ - [\Delta;\lambda]_-
\end{aligned}$$

where " signifies 'difference'. It is often convenient to introduce the symbol $\Box$ which stands for $[1^k]$ (King et al. 1981) such that

$$\begin{aligned}
\Box &= [1^k]_+ + [1^k]_- = \Box_+ + \Box_- \\
\Box'' &= [1^k]_+ - [1^k]_- = \Box_+ - \Box_- \ .
\end{aligned}$$

Upper case letters, A,B,C,$\cdots$ etc. are used for S-function series as discussed in the last chapter. Detailed properties of these series were listed in table I.3.1.

II.2 SYMMETRIZED POWERS OF BASIC SPIN IRREPS OF SO_n

The m-th Kronecker power of the basic spin irrep Δ of SO_{2k+1} and

$\Delta_\pm$ of SO_{2k} can be resolved in accordance with the reduction

$$\Delta^m = \sum_{\lambda \vdash m} f^\lambda \, \Delta \otimes \{\lambda\} \qquad \text{(II.2-1a)}$$

$$\Delta^m_\pm = \sum_{\lambda \vdash m} f^\lambda \Delta_\pm \otimes \{\lambda\} \qquad \text{(II.2-1b)}$$

where $\Delta^m = \Delta \times \Delta \times \cdots \times \Delta$ (m copies), (λ) is a partition of the integer m, f^λ is the dimension or degree of the corresponding irrep of the symmetry group S_m and $\Delta \otimes \{\lambda\}$ is the m-th *symmetrized (Kronecker) power of basic spin irrep*, or *spin plethysm*. For $\{\lambda\} = \{1^m\}$, the $\Delta \otimes \{1^m\}$ is called *antisymmetric (Kronecker) power of basic spin irrep* due to the fact that $\{1^m\}$ is completely antisymmetric.

The antisymmetric power of basic spin irreps are specially important since any other symmetrized powers may be evaluated from them by taking Kronecker products. This is due to the fact that every S-function $\{\lambda\}$ (or character of unitary group) can be expressed as a sum of products of the elementary symmetric functions $e_r = \{1^r\}$ via equation (I.2.2-7) and the properties of plethysms (Wybourne 1970, King et al. 1981):

$$\Delta \otimes (\{\lambda\} + \{\mu\}) = \Delta \otimes \{\lambda\} + \Delta \otimes \{\mu\} \qquad \text{(II.2-2a)}$$

$$\Delta \otimes (\{\lambda\} \cdot \{\mu\}) = \Delta \otimes \{\lambda\} \cdot \Delta \otimes \{\mu\} \qquad \text{(II.2-2b)}$$

The number of independent non-zero antisymmetric powers of basic spin irrep for any group SO_n is limited to be α when n=2 mod 4, $\alpha/2$ in other cases, where α is the dimension (or degree) of the basic spin irrep of SO_n:

$$\alpha = \begin{cases} 2^{k-1} & \text{if } n=2k \\ 2^k & \text{if } n=2k+1 \end{cases} \qquad \text{(II.2-3)}$$

This result can be justified from the relation between plethysms and branching rules and the property of plethysm.

Plethysms are intimately related to branching rules (Wybourne 1970) corresponding to each group-subgroup embedding defined by the decomposition of the vector irrep there is a branching rule and a plethysm. We state the theorem given by Wybourne as follows:

Let H be a subgroup of G which has a unitary irrep $\{1\}_G$ and other irreps $\{\lambda\}_G$. Let $\{\sigma\}_H$ be the unitary irrep of H such that $\dim(\sum_\sigma\{\sigma\}_H) = \dim\{1\}_G$, then the decomposition $G \downarrow H$, $\{1\}_G \downarrow \sum_\sigma\{\sigma\}_H$ defines a group-subgroup embedding under which $\{\lambda\}_G \downarrow (\sum_\sigma\{\sigma\}_H) \otimes \{\lambda\}_G$, i.e.

$$
\begin{array}{llll}
 & G & \downarrow & H \\
\text{if} & \{1\}_G & \downarrow & \sum_\sigma\{\sigma\}_H \\
\text{then} & \{\lambda\}_G & \downarrow & (\sum_\sigma\{\sigma\}_H) \otimes \{\lambda\}_G
\end{array}
\qquad
\begin{array}{r}
\\ \text{(II.2-4a)} \\ \text{(II.2-4b)}
\end{array}
$$

By applying this theorem we can see that the appropriate branching rule corresponding to spin plethysm $\Delta(\text{or } \Delta_\pm) \otimes \{1^m\}$ is

$$
\begin{array}{lll}
SU_\alpha & \downarrow & SO_n \\
\{1\} & \downarrow & \Delta(\text{or } \Delta_\pm) \\
\{1^m\} & \downarrow & \Delta(\text{or } \Delta_\pm) \otimes \{1^m\}
\end{array}
\qquad
\begin{array}{r}
\\ \text{(II.2-5a)} \\ \text{(II.2-5b)}
\end{array}
$$

where α is the dimension of basic spin irrep Δ (or $\Delta_\pm$) of SO_n as given in (II.2-3). If $\{1\} \downarrow \Delta_+$ then $\{\bar{1}\} \downarrow \Delta_-$ where $\{\bar{1}\}$ is the contragradient partner of $\{1\}$ in SU_α. i.e. $\{\bar{1}\} = \{1^{\alpha-1}\}$. From the branching rule interpretation of basic spin plethysm it is clear that

$$\Delta(\text{or } \Delta_\pm) \otimes \{1^\alpha\} = \{0\} \qquad \text{(II.2-6a)}$$

$$\Delta(\text{or } \Delta_\pm) \otimes \{1^{\alpha+x}\} = 0 \qquad \text{if } x > 0 \qquad \text{(II.2-6b)}$$

and indeed

$$\Delta(\text{or } \Delta_{\pm}) \otimes \{\lambda\} = 0 \qquad \text{if} \quad \ell(\lambda) > \alpha \qquad \text{(II.2-6c)}$$

due to the modification rule of SU_{α}:

$$\{1^{\alpha}\} = \{0\},$$
$$\{1^{\alpha+x}\} = 0, \qquad \text{if } x > 0$$
$$\{\lambda\} = 0, \qquad \text{if } \ell(\lambda) > \alpha.$$

Consider now the reality properties of basic spin irreps. For any unitary irrep T of group G, its complex conjugate T^{*} is also an unitary irrep of the group. Irreps may be classified into three types according to the relation betwen T and T^{*}. If T is equivalent to T^{*}, i.e. there exists a unitary matrix U such that

$$T = U\, T^{*}\, U^{-1}$$

then the character of irrep T is *real*, and if U is symmetric, T is *orthogonal*, if U is antisymmetric then T is *symplectic* (*pseudoreal*). On the other hand, if T is not equivalent to T^{*} then the irrep T is *complex*.

A more practical method of classifying irreps is by the analysis of symmetrized Kronecker second powers and Kronecker squares (Luan 1982) to give

$$\text{if} \quad T \otimes \{2\} \supset I \qquad T \text{ is orthogonal} \qquad \text{(II.2-7a)}$$
$$\text{if} \quad T \otimes \{1^2\} \supset I \qquad T \text{ is symplectic} \qquad \text{(II.2-7b)}$$
$$\text{if} \quad T^2 = T \times T \not\supset I \qquad T \text{ is complex} \qquad \text{(II.2-7c)}$$

where I is the identity irrep of G.

According to the above criteria the basic spin irreps Δ, $\Delta_{\pm}$ of SO_n ($n = 2k+1$ or $2k$) is *orthogonal* if

$$\Delta(\text{or } \Delta_\pm) \otimes \{2\} \supset [0] \qquad \text{(II.2-8a)}$$

symplectic if

$$\Delta(\text{or } \Delta_\pm) \otimes \{1^2\} \supset [0] \ , \qquad \text{(II.2-8b)}$$

and *complex* if

$$\Delta^2(\text{or } \Delta_\pm^2) \not\supset [0] \ , \qquad \text{(II.2-8c)}$$

where the symmetrized powers of order two and the Kronecker square of basic spin irreps are given as (King et al. 1981):

$$SO_{2k+1} : \Delta \otimes \{2\} = [1^k] + \sum_{x=0}^{k} ([1^{k-3-4x}] + [1^{k-4-4x}]), \qquad \text{(II.2-9a)}$$

$$\Delta \otimes \{1^2\} = \sum_{x=0}^{k} ([1^{k-1-4x}] + [1^{k-2-4x}]), \qquad \text{(II.2-9b)}$$

$$\Delta^2 = \sum_{x=0}^{k} [1^{k-x}], \qquad \text{(II.2-9c)}$$

$$SO_{2k} : \Delta_\pm \otimes \{2\} = [1^k]_\pm + \sum_{x=0}^{k} [1^{k-4-4x}], \qquad \text{(II.2-9d)}$$

$$\Delta_\pm \otimes \{1^2\} = \sum_{x} [1^{k-2-4x}], \qquad \text{(II.2-9e)}$$

$$\Delta_\pm^2 = [1^k]_\pm + \sum_{x=0}^{k} [1^{k-2-2x}] \ . \qquad \text{(II.2-9f)}$$

respectively. By noting that $[0] = [1^0]$ it is easy to obtain the classification of basic spin irreps of SO_n as shown in table II.2.1.

Table II.2.1 : Classification of basic spin irreps Δ (or $\Delta_\pm$) of SO_n

n		Reality type of basic spin irrep of SO_n	
0,1,3 (mod 4)	0,1,7 (mod 8)	real	orthogonal
	3,4,5 (mod 8)		symplectic
2 (mod 4)	2,6 (mod 8)	complex	

This classification enables us to further reduce the number of independent antisymmetric powers spin irreps as follows.

Consider any S-function defined on complex variables. We can establish the result

$$\{\lambda\} \otimes \{\mu\}^{*} = (\{\lambda\} \otimes \{\mu\})^{*} \qquad \text{(II.2-10)}$$

quite readily since plethysm can be interpreted as a process of substitution and the complex conjugation has the properties $(x\ y)^{*} = x^{*}y^{*}$, $(x+y)^{*} = x^{*} + y^{*}$ where $x, y \in \mathbb{C}$.

The contragradient partners $\{1^m\}$ and $\overline{\{1^m\}}$ of SU_α are related by

$$\overline{\{1^m\}} = \{1^{\alpha-m}\}, \qquad \overline{\{1^m\}} = \{1^m\}^{*} \qquad \text{(II.2-11)}$$

Thus we have

$$\begin{aligned} &\Delta \text{ (or } \Delta_{\pm}) \otimes \{1^m\} \\ &= \Delta \text{ (or } \Delta_{\pm}) \otimes \{1^{\alpha-m}\}^{*} \\ &= (\Delta \text{ (or } \Delta_{\pm}) \otimes \{1^{\alpha-m}\})^{*} \end{aligned} \qquad \text{(II.2-12)}$$

For $n = 0,1,3 \pmod 4$, the characters of Δ $(\Delta_{\pm})$ are real so the spin plethysm contains only real content thus

For SO_n $n = 0,1,3 \pmod 4$:

$$\Delta(\text{or } \Delta_{\pm}) \otimes \{1^m\} = \Delta(\text{or } \Delta_{\pm}) \otimes \{1^{\alpha-m}\} \qquad \text{(II.2-13)}$$

For $n = 2 \pmod 8$, the characters of $\Delta_{\pm}$ are complex hence the spin plethysm contains both real and complex content. It is known that in this case

$[\lambda]$ with $\lambda_k = 0$ is real, $[\lambda]_{\pm}^{*} = [\lambda]_{\mp}$, $(\Delta_{\pm})^{*} = \Delta_{\mp}$, $([\Delta;\lambda]_{\mp})^{*} = [\Delta;\lambda]_{\mp}$.

Thus

For SO_n, n = 2 mod 4:

$$\Delta_{\pm} \otimes \{1^m\} = (\Delta\pm\otimes\{1^{\alpha-m}\})^{\dagger} \qquad \text{(II.2-14)}$$

where the † is the *involutary outer automorphism of* SO_{2k} (Black et al. 1983)

$$[\lambda]^{\dagger} = [\lambda] \quad \text{with } \lambda_k = 0, \quad n = 2k \qquad \text{(II.2-15a)}$$

$$([\lambda]_{\pm})^{\dagger} = [\lambda]_{\mp} \qquad \text{(II.2-15b)}$$

$$(\Delta_{\pm})^{\dagger} = \Delta_{\mp} \qquad \text{(II.2-15c)}$$

$$([\Delta;\lambda]_{\pm})^{\dagger} = [\Delta;\lambda]_{\mp} \qquad \text{(II.2-15d)}$$

Using results (II.2-6), (II.2-13) and (II.2-14) together with (II.2-2) we need only to find Δ(or $\Delta_{\pm}$) $\otimes$ $\{1^m\}$ up to power $\alpha/2$ in order to find all other symmetrized powers of basic spin irreps of SO_n.

Properties of local isomorphism and outer automorphism of SO_n play an important part in evaluating spin plethysms (Littlewood 1950). Under the isomorphic mapping ~,spin irrep Δ (or $\Delta_{\pm}$) becomes an tensor irrep in the corresponding group in which the plethysm may be easily evaluated. Application of the inverse mapping ~' (~ ~' = I, the identity) then convert the result back to the original group. Properties of isomorphism of SO_n for n = 2,3,4,5,6 and automorphism for SO_8 are listed in table II.2.2. The plethysms of tensor irreps of SO_n and Sp_{2k} can be evaluated using the formula (King et al. 1981)

$$Sp_{2k}: \quad \langle\lambda\rangle \otimes \{\mu\} = \langle(\{\lambda/A\} \otimes \{\mu\})/B\rangle \qquad \text{(II.2-16a)}$$

$$O_n \quad : \quad [\lambda] \otimes \{\mu\} = [(\{\lambda/C\} \otimes \{\mu\})/D] \qquad \text{(II.2-16b)}$$

where the second formula also holds for SO_{2k+1} and SO_{2k} if $\lambda_k \neq 0$.

Example 1: Evaluate $\Delta \otimes \{1^3\}$ in SO_5. From table II.2.1, SO_5 is

Table II.2.2 : Local isomorphism of SO_n for n = 2,3,4,5,6 and outer automorphism of SO_8

Isomorphic groups	Relation between irreps under isomorphic mapping ~	Basic spin irreps under ~
$SO_2 \sim U_1$	$[a] \sim \{a\}$	$\Delta_+ \sim \{\frac{1}{2}\}$ $\Delta_- \sim \{\frac{1}{2}\}$
$SO_3 \sim SU_2$	$[a] \sim \{2a\}$	$\Delta \sim [1]$
$SO_4 \sim SU_2 \times SU_2$	$[a\ b] \sim \{a+b\} \times \{a-b\}$	$\Delta_+ \sim \{1\} \times \{0\}$ $\Delta_- \sim \{0\} \times \{1\}$
$SO_5 \sim Sp_4$	$[a\ b] \sim \langle a+b\ a-b\rangle$	$\Delta \sim \langle 1\rangle$
$SO_6 \sim SU_4$	$[a\ b\ c] \sim \{a+b\ a-c\ b-c\}$	$\Delta_+ \sim \{1\}$ $\Delta_- \sim \{1^3\}$
$SO_8 \sim SO_8$	$[a\ b\ c\ d] \sim$ $[\frac{a+b+c+d}{2}\ \frac{a+b-c-d}{2}\ \frac{a-b+c-d}{2}\ \frac{-a+b+c-d}{2}]$	$\Delta_+ \sim [1]$ $\Delta_- \sim \Delta_+$

Isomorphic groups	Relation between irreps under isomorphic mapping ~'	
$U_1 \sim' SO_2$	$\{a\} \sim' [a]$	$\equiv [\{a\}]'$
$SU_2 \sim' SO_3$	$\{a\} \sim' [\frac{a}{2}]$	$\equiv [\{a\}]'$
$SU_2 \times SU_2 \sim SO_4$	$\{a\} \times \{b\} \sim' [\frac{a+b}{2}\ \frac{a-b}{2}]$	$\equiv [\{a\} \times \{b\}]'$
$Sp_4 \sim' SO_5$	$\langle a\ b\rangle \sim' [\frac{a+b}{2}\ \frac{a-b}{2}]$	$\equiv [\langle a\ b\rangle]'$
$SU_4 \sim' SO_6$	$\{a\ b\ c\} \sim' [\frac{a+b-c}{2}\ \frac{a-b+c}{2}\ \frac{a-b-c}{2}]$	$\equiv [\{a\ b\ c\}]'$
$SO_8 \sim' SO_8$	$[a\ b\ c\ d] \sim' [\frac{a+b+c+d}{2}\ \frac{a+b-c+d}{2}\ \frac{a-b+c+d}{2}\ \frac{a-b-c-d}{2}]$	$\equiv [[a\ b\ c\ d]]'$

Note: $\sim \sim' = \sim' \sim$ = identity. Symbol [ ]' stands for isomorphic mapping ~', which yields an irrep of SO_n when applied to an irrep of a group, locally isomorphic to SO_n, which is enclosed in the bracket.

isomorphic to Sp_4 locally, and $\Delta \sim \langle 1 \rangle$. Thus

$$SO_5 \sim Sp_4 \sim' SO_5$$

$$\Delta \sim \langle 1 \rangle \sim' \Delta$$

$$\begin{aligned} \Delta \otimes \{1^3\} &\sim \langle 1 \rangle \otimes \{1^3\} \\ &= \langle 1^3/B \rangle && \text{by (II.2-16a)} \\ &= \langle 1^3 \rangle + \langle 1 \rangle \\ &= 0 + \langle 1 \rangle \\ &\sim' \Delta \end{aligned}$$

$$\therefore \quad \Delta \otimes \{1^3\} = \Delta \qquad \text{in } SO_5$$

Note that the modification rule (Black et al. 1983)

$$Sp_{2k}: \langle \lambda \rangle = (-1)^x \langle \lambda - h \rangle, \quad h = 2p - 2k - 2$$

has been applied.

Example 2: Evaluate $\Delta_+ \otimes \{1^4\}$ in SO_8. From table II.2.1,

$$SO_8 \sim SO_8 \sim' SO_8$$

$$\Delta_+ \sim [1] \sim' \Delta_+$$

$$\begin{aligned} \Delta_+ \otimes \{1^4\} &\sim [1] \otimes \{1^4\} \\ &= [1^4/D] && \text{by (II.2-16b)} \\ &= [1^4]_+ + [1^4]_- \\ &\sim' [1^4]_- + [2] \end{aligned}$$

$$\therefore \Delta_+ \otimes \{1^4\} = [1^4]_- + [2] \qquad \text{in } SO_8$$

Simple formulas for calculating spin plethysm of SO_n may be readily obtained for $n \leq 9$. For those SO_n where isomorphism (or automorphism) exists, use are made of the results of table II.2.2 and equation (II.2-16a,b). For SO_7 and SO_9 branching rules must be used together with the formulas for SO_6 and SO_8 respectively to derive the results. For example, to calculate $\Delta \otimes \{\lambda\}$ in SO_7, we use the branching rules (Black et al. 1983)

$$SO_7 \downarrow SO_6$$

$$\Delta \downarrow \Delta_+ + \Delta_- \qquad \text{(II.2-17a)}$$

$$\Delta \otimes \{\lambda\} \downarrow (\Delta_+ + \Delta_-) \otimes \{\lambda\} \qquad \text{(II.2-17b)}$$

and

$$SO_6 \uparrow SO_7$$

$$[\lambda] \uparrow [\lambda]/L \equiv [\lambda/L] \qquad \text{(II.2-17c)}$$

$$[\Delta;\lambda] \uparrow [\Delta;\lambda]/L \equiv [\Delta;\lambda/L] \qquad \text{(II.2-17d)}$$

$$\Delta \uparrow \Delta/L = \Delta \qquad \text{(II.2-17e)}$$

where $[\lambda]$ with $\lambda_3 \neq 0$, $[\Delta;\lambda]$ and Δ are 'sum characters' of SO_6, i.e. $[\lambda] = [\lambda]_+ + [\lambda]_-$, $[\Delta;\lambda] = [\Delta;\lambda]_+ + [\Delta;\lambda]_-$, $\Delta = \Delta_+ + \Delta_-$, together with the property of plethysm (Wybourne 1970)

$$(A + B) \otimes \{\sigma\} = \sum_\rho A \otimes \{\sigma/\rho\} \cdot B \otimes \{\rho\} \qquad \text{(II.2-18)}$$

and the property of involutary automorphism † (Black et al. 1981)

$$(\Delta_+ \otimes \{\lambda\})^\dagger = \Delta_- \otimes \{\lambda\} \qquad \text{(II.2-19)}$$

we arrive at

$$SO_7: \quad \Delta \otimes \{\lambda\} = \sum_\rho [([\{\lambda/\rho\}]'_{SO_6} \cdot [\{\rho\}]'^{\dagger}_{SO_6})/L] \qquad \text{(II.2-20)}$$

where the skewing of SO_6 irreps with L series is as defined in (II.2-17c, d,e).

An alternative way is to consider the branching rules (Black et al. 1983)

$$SO_7 \uparrow SO_8$$

$$\Delta \uparrow \Delta_+ \qquad \text{(II.2-21a)}$$

$$\Delta \otimes \{\lambda\} \uparrow \Delta_+ \otimes \{\lambda\} \qquad \text{(II.2-21b)}$$

and

$$SO_8 \downarrow SO_7$$

$$[\lambda] \downarrow [\lambda]/M \equiv [\lambda/M] \tag{II.2-21c}$$

$$[\lambda]_{\pm} \downarrow [\lambda]_{\pm}/M \equiv \tfrac{1}{2}[\lambda/M] \tag{II.2-21d}$$

$$[\Delta;\lambda]_{\pm} \downarrow [\Delta;\lambda]_{\pm}/M = \begin{cases} [\Delta;\lambda/M] & \text{if } \lambda_4 = 0 \quad \text{(II.2-21e)} \\ \tfrac{1}{2}[\Delta;\lambda/M] & \text{if } \lambda_4 \neq 0 \quad \text{(II.2-21f)} \end{cases}$$

$$\Delta_{\pm} \downarrow \Delta/M \equiv \Delta \tag{II.2-21g}$$

to arrive at

$$SO_7\text{: } \Delta \otimes \{\lambda\} = [[\lambda/D]]'_{SO_8}/M \tag{II.2-22}$$

where the skewing of SO_8 irrep with M series is as defined in (II.2-21 c-g), $[\ ]'_{SO_8}$ is the automorphic mapping $\sim'$ of SO_8.

Spin plethysm for SO_{10} have been given by Black et al. (1983) but no general formula could be found. SO_{11} result depends on SO_{10} in the same fashion as SO_7 on SO_6, and SO_9 on SO_8. In fact, if the SO_{2k} spin plethysm is known, then the result for SO_{2k+1} would follow immediately.

In table II.2.3, we have tabulated the formulas of symmetrized power of basic spin irreps of SO_n for $n \leq 9$. Antisymmetric and symmetrized power of basic spin irrep of SO_n for $n \leq 10$ up to power $\alpha/2$ and 4 respectively, have been collected in table II.2.4 and II.2.5. Table II.2.4 also includes the results for SO_{11} up to power 8. Since $\Delta_- \otimes \{\lambda\}$ is related to $\Delta_+ \otimes \{\lambda\}$ by the involutary outer automorphism † as given in (II.2-19), it is not listed separately.

The classification and calculation of basic spin irreps are very important in determining the automorphism group K and in obtaining the relevant branching rules for extended Poincaré supersymmetry.

II.3 BRANCHING RULES FOR SUBGROUPS OF SO_{2k}

We will use the well developed S-function techniques to derive several branching rules for the special orthogonal group SO_{2k} for the

Table II.2.3 : Formulas for calculating spin plethysms of SO_n for $n \leq 9$

Group	Spin Plethysm
SO_n (n = 2,3,6)	$\Delta(\text{or } \Delta_+) \otimes \{\lambda\} = [\{\lambda\}]'$
SO_4	$\Delta_+ \otimes \{\lambda\} = [\{\lambda\} \times \{0\}]'$
SO_5	$\Delta \otimes \{\lambda\} = [\langle\lambda/B\rangle]'$
SO_8	$\Delta_+ \otimes \{\lambda\} = [[\lambda/D]]'$
SO_7	$\Delta \otimes \{\lambda\} = [[[\lambda/D]]'_{SO_8} /M]$
	$\Delta \otimes \{\lambda\} = \sum_\rho[([\{\lambda/\rho\}]'_{SO_6} \cdot [\{\rho\}]'^{\dagger}_{SO_6})/L]$
SO_9	$\Delta \otimes \{\lambda\} = \sum_\rho[([[\lambda/\rho D]]'_{SO_8} \cdot [[\rho/D]]'^{\dagger}_{SO_8})/L]$

Note: B,D,L,M are S-function series, [ ]' is the isomorphic mapping ~' as given in table II.2.2, † is the involuntary outer automorphism defined in (II.2-15).

Table II.2.4 : Antisymmetric m-th powers of the basic spin irreps of SO_n ($n \leq 11$, $m \leq 8$)

m	$\Delta_+ \otimes \{1^m\}$	SO_2	SO_4	SO_6	SO_8	SO_{10}
1	$\Delta_+ \otimes \{1\}$	Δ_+	Δ_+	Δ_+	Δ_+	$\Delta+$
2	$\Delta_+ \otimes \{1^2\}$		$[0]$	$[1]$	$[1^2]$	$[1^3]$
3	$\Delta_+ \otimes \{1^3\}$			Δ_-	$[\Delta,1]_-$	$[\Delta;1^2]_-$
4	$\Delta_+ \otimes \{1^4\}$			$[0]$	$[2]+[1^4]_-$	$[2^2]+[21^4]_-$
5	$\Delta_+ \otimes \{1^5\}$				$[\Delta;1]_-$	$[\Delta;21]_-+[\Delta;1^5]_-$
6	$\Delta_+ \otimes \{1^6\}$				$[1^2]$	$[2^21^3]_-+[31^2]$
7	$\Delta_+ \otimes \{1^7\}$				Δ_+	$[\Delta;21^2]_-+[\Delta;3]_+$
8	$\Delta_+ \otimes \{1^8\}$				$[0]$	$[4]+[2^3]+[31^3]$

m	$\Delta \otimes \{1^m\}$	SO_3	SO_5	SO_7	SO_9
1	$\Delta \otimes \{1\}$	Δ	Δ	Δ	Δ
2	$\Delta \otimes \{1^2\}$	$[0]$	$[1]+[0]$	$[1^2]+[1]$	$[1^3]+[1^2]$
3	$\Delta \otimes \{1^3\}$		Δ	$[\Delta;1]+\Delta$	$[\Delta;1^2]+[\Delta;1]$
4	$\Delta \otimes \{1^4\}$		$[0]$	$[2]+[1^3]+[1]+[0]$	$[21^3]+[1^4]+[2^2]+[21]+[2]$
5	$\Delta \otimes \{1^5\}$			$[\Delta;1]+\Delta$	$[\Delta;21]+[\Delta;2]+[\Delta;1^4]+[\Delta;1^2]+[\Delta;1]$
6	$\Delta \otimes \{1^6\}$			$[1^2]+[1]$	$[2^21^2]+[21^3]+[31^2]+[31]+[21^2]+[21]+[1^3]+[1^2]$
7	$\Delta \otimes \{1^7\}$			Δ	$[\Delta;3]+[\Delta;21^2]+[\Delta;21]+[\Delta;2]+[\Delta;1^3]+[\Delta;1^2]+[\Delta;1]+\Delta$
8	$\Delta \otimes \{1^8\}$			$[0]$	$[4]+[31^3]+[31^2]+[3]+[2^3]+[2^21]+[2^2]+[21^3]+[21^2]+[2]+[1^4]+[1^3]$ $+[1]+[0]$

Table II.2.4 continued on next page

Table II.2.4 continued

m	$\Delta \oplus \{1^m\}$	SO_{11}
1	$\Delta \otimes \{1\}$	Δ
2	$\Delta \otimes \{1^2\}$	$[1^4]+[1^3]+[0]$
3	$\Delta \otimes \{1^3\}$	$[\Delta;1^3]+[\Delta;1^2]+\Delta$
4	$\Delta \otimes \{1^4\}$	$[2^3]+[2^21^3]+[2^21]+[2^2]+[21^4]+[1^4]+[1^3]+[0]$
5	$\Delta \otimes \{1^5\}$	$[\Delta;2^21]+[\Delta;2^2]+[\Delta;21^4]+[\Delta;21^2]+[\Delta;21]+[\Delta;1^5]+[\Delta;1^3]+[\Delta;1^2]+\Delta$
6	$\Delta \otimes \{1^6\}$	$[3^21^2]+[3^31]+[32^21^2]+[321^3]+[321^2]+[321]+[31^3]+[31^2]+[2^5]+[2^31^2]+[2^3]+2[2^21^3]+[2^21]+[2^2]+[21^4]+[1^4]$ $+[1^3]+[0]$
7	$\Delta \otimes \{1^7\}$	$[\Delta;3^2]+[\Delta;321^2]+[\Delta;321]+[\Delta;32]+[\Delta;31^3]+[\Delta;31^2]+[\Delta;31]+[\Delta;3]+[\Delta;2^31^2]+[\Delta;2^21^3]+[\Delta;2^21^2]+2[\Delta;2^21]+[\Delta;2^2]$ $+[\Delta;21^4]+[\Delta;21^3]+2[\Delta;21^2]+[\Delta;21]+[\Delta;1^5]+[\Delta;1^3]+[\Delta;1^2]+\Delta$
8	$\Delta \otimes \{1^8\}$	$[4^2]+[431^3]+[431^2]+[43]+[42^3]+[42^21]+[42^2]+[421^3]+[421^2]+[42]+[41^4]+[41^3]+[41]+[4]+[3^22^21]+[3^221^2]$ $+[3^21^3]+2[3^21^2]+[3^21]+[32^31]+[32^3]+2[32^21^2]+[32^21]+[32^2]+2[321^3]+2[321^2]+[321]+[31^4]+2[31^3]+[31^2]+[2^5]$ $+[2^4]+[2^31^2]+[2^31]+2[2^3]+2[2^21^3]+[2^21]+[2^2]+[21^4]+[1^4]+[1^3]+[0]$

Note: for SO_9, $\Delta \otimes \{1^m\} = \Delta \otimes \{^{16-m}\}$, for SO_{10}, $\Delta \otimes \{1^m\} = (\Delta \otimes \{1^{16-m}\})^\dagger$,
for SO_{11}, $\Delta \otimes \{1^m\} = \Delta \otimes \{1^{32-m}\}$, for SO_{2k}, $\Delta_- \otimes \{1^m\} = (\Delta_+ \otimes \{1^m\})^\dagger$,
where $\dagger$ is the involutary outer automorphism defined in (II.2-15).

Table II.2.5 : Symmetrized m-th powers of basic spin irreps of SO_n $(n\leq 10, m\leq 4)$

M	$\Delta_+\otimes\{\lambda\}$	SO_4	SO_6	SO_8	SO_{10}
2	$\Delta_+\otimes\{2\}$	$[1^2]_+$	$[1^3]_+$	$[1^4]_+ + [0]$	$[1^5]_+ + [1]$
	$\Delta_+\otimes\{1^2\}$	$[0]$	$[1]$	$[1^2]$	$[1^3]$
3	$\Delta_+\otimes\{3\}$	$[\Delta;1]_+$	$[\Delta;1^3]_+$	$[\Delta;1^4]_+ + \Delta_+$	$[\Delta;1^5]_+ + [\Delta;1]_+$
	$\Delta_+\otimes\{21\}$	Δ_+	$[\Delta;1]_+$	$[\Delta;1^2]_+ + \Delta_+$	$[\Delta;1^3]_+ + [\Delta;1]_+ + \Delta_-$
	$\Delta_+\otimes\{1^3\}$	-	Δ_-	$[\Delta;1]_-$	$[\Delta;1^2]_-$
4	$\Delta_+\otimes\{4\}$	$[2^2]_+$	$[2^3]_+$	$[2^4]_+ + [1^4]_+ + [0]$	$[2^5]_+ + [21^4]_+ + [2]$
	$\Delta_+\otimes\{31\}$	$[1^2]_+$	$[21^2]_+$	$[2^2 1^2]_+ + [1^4]_+ + [1^2]$	$[2^3 1^2]_+ + [21^4]_+ + [21^2] + [1^4]_+$
	$\Delta_+\otimes\{2^2\}$	$[0]$	$[2]$	$[2^2] + [1^4]_+ + [0]$	$[2^3] + [21^4]_+ + [2] + [1^4] + [0]$
	$\Delta_+\otimes\{21^2\}$	-	$[1^2]$	$[21^2] + [1^2]$	$[2^2 1^2] + [21^2] + [1^4] + [1^2]$
	$\Delta_+\otimes\{1^4\}$	-	$[0]$	$[2] + [1^4]_-$	$[2^2] + [21^4]_-$

Table II.2.5 continued on next page

Table II.2.5 continued

m	$\Delta\otimes\{\lambda\}$	SO_3	SO_5	SO_7	SO_9
2	$\Delta\otimes\{2\}$	$[1]$	$[1^2]$	$[1^3]+[0]$	$[1^4]+[1]+[0]$
	$\Delta\otimes\{1^2\}$	$[0]$	$[1]+[0]$	$[1^2]+[1]$	$[1^2]+[1]$
3	$\Delta\otimes\{3\}$	$[\Delta;1]$	$[\Delta;1^2]$	$[\Delta;1^3]+\Delta$	$[\Delta;1^4]+[\Delta;1]+\Delta$
	$\Delta\otimes\{21\}$	Δ	$[\Delta;1]+\Delta$	$[\Delta;1^2]+[\Delta;1]+\Delta$	$[\Delta;1^3]+[\Delta;1^2]+[\Delta;1]+2\Delta$
	$\Delta\otimes\{1^3\}$	–	Δ	$[\Delta;1]+\Delta$	$[\Delta;1^2]+[\Delta;1]$
4	$\Delta\otimes\{4\}$	$[2]$	$[2^2]$	$[2^3]+[1^3]+[0]$	$[2^4]+[21^3]+[2]+[1^4]+[1]+[0]$
	$\Delta\otimes\{31\}$	$[1]$	$[21]+[1^2]$	$[2^21]+[21^2]+[1^3]+[1^2]+[1]$	$[2^31]+[2^21^2]+[21^3]+[21^2]+[21]+2[1^4]+2[1^3]+2[1^2]+[1]$
	$\Delta\otimes\{2^2\}$	$[0]$	$[2]+[1]+[0]$	$[2^2]+[21]+[2]+[1^3]+[0]$	$[2^3]+[2^21]+[2^2]+[21^3]+[2]+2[1^4]+[1^3]+[1]+2[0]$
	$\Delta\otimes\{21^2\}$	–	$[1^2]+[1]$	$[21^2]+[21]+[1^3]+2[1^2]+[1]$	$[2^21^2]+[2^21]+[21^3]+2[21^2]+[21]+[1^4]+2[1^3]+2[1^2]+[1]$
	$\Delta\otimes\{1^4\}$	–	$[0]$	$[2]+[1^3]+[1]+[0]$	$[2^2]+[21^3]+[21]+[2]+[1^4]$

group-subgroup combinations

$$SO_{4rs} \supset SO_{2r} \times SO_{2s}$$
$$SO_{4rs+2s} \supset SO_{2r+1} \times SO_{2s}$$
$$SO_{4rs+2r+2s+1} \supset SO_{2r+1} \times SO_{2s+1}$$
$$SO_{4rs} \supset Sp_{2r} \times Sp_{2s}$$

The embedding is determined by the specified decomposition of the vector irrep [1] as given in table II.3.1. The decompositions of spin irrep Δ, Δ'' and tensor irrep $[\lambda]$ of SO_n have already been found by Morris (1958,1961), Wybourne (1981) and King (1975). Using their results together with the Kronecker product formulae (Black, King, Wybourne 1983)

For SO_n, n even or odd:

$$[\lambda]\cdot[\mu] = \sum_{\zeta}[(\lambda/\zeta)\cdot(\mu/\zeta)] \tag{II.3-1}$$

$$[\lambda]\cdot[\Delta;\mu] = \sum_{\zeta}[\Delta;\ (\lambda/\zeta Q)\cdot(\mu/\zeta)] \tag{II.3-2}$$

$$[\Delta;\lambda]''\cdot[\mu] = \sum_{\zeta}[\Delta;(\lambda/\zeta)\cdot(\mu/\zeta L)]'' \tag{II.3-3}$$

$$[\Delta;\lambda]_{\pm}\cdot[\mu] = \sum_{\zeta,n}[\Delta;(\lambda/\zeta)\cdot(\mu/\zeta 1^{n})]_{\pm(-)^{n}} \tag{II.3-4}$$

$$[\Delta;\lambda]''\cdot[\Delta;\mu] = \sum_{\zeta}[\Box;(\lambda/\zeta Q)\cdot(\mu/\zeta L)]'' \tag{II.3-5}$$

$$[\Box;\lambda]''\cdot[\mu] = \sum_{\zeta}[\Box;(\lambda/\zeta)\cdot(\mu/\zeta V)]'' \tag{II.3-6}$$

For Sp_{2k}:

$$\langle\lambda\rangle\cdot\langle\mu\rangle = \sum_{\zeta}\langle(\lambda/\zeta)\cdot(\mu/\zeta)\rangle \tag{II.3-7}$$

and the identities (King 1975, 1981, Black, King and Wybourne 1983)

For SO_n, n=2k or 2k+1:

$$[\Delta;\lambda] = \Delta[\lambda/P] \tag{II.3-8}$$

$$[\Delta;\lambda]'' = \Delta''[\lambda/M] \tag{II.3-9}$$

$$[\Delta,\lambda]_{\pm} = \sum_{m}(-)^{m}\Delta_{\pm(-)^{m}}[\lambda/m] \qquad \text{(II.3-10)}$$

$$[\square;\lambda]'' = \square''[\lambda/W] \qquad \text{(II.3-11)}$$

$$\square'' = \Delta\Delta'' \qquad \text{(II.3-12)}$$

$$\square_{\pm} = \Delta_{\pm}^{2} - \sum_{p}[1^{k-2-2p}], \qquad \text{(II.3-13)}$$

we are able to derive the branching rules for the other irreps of SO_n. The results are collected together in table II.3.1.

In deriving these results extensive use of the S-function series (Table I.3.1) has been made and particularly the following S-function series identities (King et al. 1981; Black and Wybourne 1983)

$$A = CW, \; B = DV, \; E = PC, \; F = QD, \; G = MC, \; H = DL$$

have been used.

The results for the decomposition of the basic spin irreps are of crucial importance in establishing our subsequent results for branching rules for the basic spin irreps of $SO_{2k} \downarrow SO_{D-2} \times K$. For some special values of r and s, the results are listed in table II.3.2. We note that in establishing the second result in this table i.e. for $SO_{4(n_{+}+n_{-})}$, the following branching formula

$$SO_{4(n_{+}+n_{-})} \downarrow SO_{4n_{+}} \times SO_{4n_{-}}$$

$$\Delta_{\pm} \downarrow \Delta_{+} \times \Delta_{\pm} + \Delta_{-} \times \Delta_{\mp}$$

has been used which was given by Black and Wybourne (1983).

II.4 BRANCHING RULES FOR THE EXTENDED D-DIMENSIONAL POINCARÉ SUPERSYMMETRY

In this section we derive the branching rules for the basic spin

Table II.3.1 : Branching rules for even dimensional rotation group SO_{2k}

(a) $SO(4rs) \downarrow Sp(2r) \times Sp(2s)$

$[1]$	$\downarrow \langle 1\rangle \times \langle 1\rangle$
Δ	$\downarrow \sum_{\zeta}\langle s^r/\zeta\rangle \times \langle\zeta'\rangle$
Δ''	$\downarrow \sum_{\zeta}(-1)^{\omega_\zeta}\langle s^r/\zeta\rangle \times \langle\zeta'\rangle$
$\Delta_\pm$	$\downarrow \sum_{\zeta_\pm}\langle s^r/\zeta_\pm\rangle \times \langle\zeta'_\pm\rangle$
$[\lambda]$	$\downarrow \sum_{\eta}\langle((\lambda/C)\circ\eta)/B\rangle \times \langle\eta/B\rangle$
$\Box$	$\downarrow \sum_{\eta\,\vdash\,2rs}\langle\eta/B\rangle \times \langle\eta'/B\rangle$
$\Box''$	$\downarrow \sum_{\zeta,\eta,\sigma,\tau}(-1)^{\omega_\zeta}\langle(s^r/\zeta\sigma)\cdot(s^r/\eta\sigma)\rangle \times \langle(\zeta'/\tau)\cdot(\eta'/\tau)\rangle$
$\Box_\pm$	$\downarrow \sum_{\zeta_\pm,\eta_\pm,\sigma,\tau}\langle(s^r/\zeta_\pm\sigma)\cdot(s^r/\eta_\pm\sigma)\rangle \times \langle(\zeta_\pm'/\tau)\cdot(\eta_\pm'/\tau)\rangle$
	$-\sum_{\eta\,\vdash\,2rs-2-m,m}\langle\eta/B\rangle \times \langle\eta'/B\rangle$
$[\Delta;\lambda]$	$\downarrow \sum_{\zeta,\eta,\sigma,\tau}\langle(s^r/\zeta\sigma)\cdot(((\lambda/E)\circ\eta)/B\sigma)\rangle \times \langle(\zeta'/\tau)\cdot(\eta/B\tau)\rangle$
$[\Delta;\lambda]''$	$\downarrow \sum_{\zeta,\eta,\sigma,\tau}(-1)^{\omega_\zeta}\langle(s^r/\zeta\sigma)\cdot(((\lambda/G)\circ\eta)/B\sigma)\rangle \times \langle(\zeta'/\tau)\cdot(\eta/B\tau)\rangle$
$[\Delta;\lambda]_\pm$	$\downarrow \sum_{m,\eta,\sigma,\tau,\zeta_{\pm(-)^m}}(-1)^m\langle(s^r/\sigma\zeta_{\pm(-)^m})\cdot(((\lambda/Cm)\circ\eta/)\sigma B)\rangle$
	$\times\ \langle(\zeta_{\pm(-)^m}/\tau)\cdot(\eta/B\tau)\rangle$
$[\Box,\lambda]''$	$\downarrow \sum_{\zeta,\eta,\rho,\sigma,\tau,\upsilon.\theta}(-1)^{\omega_\eta}\langle(((s^r/\zeta\sigma)\cdot(s^r/\eta\sigma))/\upsilon)\cdot(((\lambda/A)\circ\rho)/B\upsilon)\rangle$
	$\times\ \langle(((\zeta'/\tau)\cdot(\eta'/\tau))/\theta)\cdot(\rho/B\theta)\rangle$

Table II.3.1 continued

(b) $SO(4rs+2s) \downarrow SO(2r+1) \times SO(2s)$

$[1]$	$\downarrow [1] \times [1]$
Δ	$\downarrow \sum_{\zeta} [s^r/\zeta] \times [\Delta;\zeta']$
Δ''	$\downarrow \sum_{\zeta} (-1)^{\omega_\zeta} [s^r/\zeta] \times [\Delta;\zeta']''$
$\Delta_\pm$	$\downarrow \sum_{\zeta} [s^r/\zeta] \times [\Delta;\zeta']_{\pm(-)^{\omega_\zeta}}$
$[\lambda]$	$\downarrow \sum_{\eta} [((\lambda/C)\circ\eta)/D] \times [\eta/D]$
$\Box$	$\downarrow \sum_{\eta \vdash 2rs+s} [\eta/D] \times [\eta'/D]$
$\Box''$	$\downarrow \sum_{m,\zeta,\eta,\sigma,\tau} (-1)^{\omega_\zeta} [s^r/\eta\sigma)\cdot(s^r/\zeta\sigma)] \times [\Box;(\eta'/\tau Q)\cdot(\zeta'/\tau L]''$
$\Box_\pm$	$\downarrow \sum_{\zeta,\eta,\sigma,\rho,\tau} [(s^r/\zeta\sigma)\cdot(s^r/\eta\sigma)]$
	$\times [\Box;(\zeta'/\rho\tau B)\circ(\eta'/\rho)\cdot Q_{\pm(-)^{\omega_\zeta+s}}]_{(-)^{\omega_\eta}}$
	$-\sum_{m,\lambda \vdash 2rs+s-2-2m} [\lambda/D] \times [\lambda'/D]$
$[\Delta;\lambda]$	$\downarrow \sum_{\zeta,\eta,\sigma,\tau,\theta} [((s^r/\zeta\sigma)\cdot(((\lambda/E)\cdot\eta)/D\sigma)] \times [\Delta;(\zeta'/\theta)\cdot(\eta/F\theta)]$
$[\Delta;\lambda]''$	$\downarrow \sum_{\zeta,\eta,\sigma,\tau,\theta} (-1)^{\omega_\zeta} [(s^r/\zeta\sigma)\cdot(((\lambda/G)\circ\eta)/D\sigma)]''$
	$\times [\Delta;(\zeta'/\theta)\cdot(\eta/H\theta)]''$
$[\Delta;\lambda]_\pm$	$\downarrow \sum_{m,n,\zeta,\eta,\sigma,\tau,\theta} (-1)^m [(s^r/\zeta\sigma)\cdot((\lambda/Cm)\circ\eta)/D\sigma]$
	$\times [\Delta;(\zeta'/\theta)\cdot(\eta/D1^n\theta)]_{\pm(-)^{\omega_\zeta+m+n}}$
$[\Box;\lambda]''$	$\downarrow \sum_{\zeta,\eta,\sigma,\tau,\theta,\rho} (-)^{\omega_\zeta} [(((s^r/\eta\sigma)\cdot(s^r/\zeta\sigma))/\theta)\cdot(((\lambda/A)\circ\eta)/D\theta)]$
	$\times [\Box;(((\eta'/\tau Q)\cdot(\zeta'/\tau L)/\rho)\cdot(\eta/B\rho)]''$

Table II 3.1 continued

(c) $SO(4rs) \downarrow SO(2r) \times SO(2s)$

$[1]$	$\downarrow [1] \times [1]$
Δ	$\downarrow \sum_{\zeta} [s^r/\zeta] \times [\zeta']$
Δ''	$\downarrow \sum_{\zeta} (-1)^{\omega_\zeta} [s^r/\zeta] \times [\zeta']$
$\Delta_\pm$	$\downarrow \sum_{\zeta_\pm} [s^r/\zeta_\pm] \times [\zeta'_\pm]$
$[\lambda]$	$\downarrow \sum_{\eta} [((\lambda/C)\circ\eta)/D] \times [\eta/D]$
$\square$	$\downarrow \sum_{\eta \vdash 2rs} [\eta/D] \times [\eta'/D]$
$\square''$	$\downarrow \sum_{\zeta,\eta,\sigma,\tau} (-1)^{\omega_\eta} [(s^r/\zeta\sigma)\cdot(s^r/\eta\sigma)] \times [(\zeta'/\tau)\cdot(\eta'/\tau)]$
$\square_\pm$	$\downarrow \sum_{\zeta_\pm,\eta_\pm,\sigma,\tau} [(s^r/\zeta_\pm\sigma)\cdot(s^r/\eta_\pm\sigma)] \times [(\zeta'_\pm/\tau)\cdot(\eta'_\pm/\tau)]$
	$-\sum_{\eta \vdash 2rs-2-m,m} [\eta/D] \times [\eta'/D]$
$[\Delta;\lambda]$	$\downarrow \sum_{\zeta,\eta,\sigma,\tau} [(s^r/\zeta\sigma)\cdot(((\lambda/E)\circ\eta)/D\sigma)] \times [(\zeta'/\tau)\cdot(\eta/D\tau)]$
$[\Delta;\lambda]''$	$\downarrow \sum_{\zeta,\eta,\sigma,\tau} (-1)^{\omega_\zeta} [(s^r/\zeta\sigma)\cdot(((\lambda/G)\circ\eta)/D\sigma)] \times [(\zeta'/\tau)\cdot(\eta/D\tau)]$
$[\Delta;\lambda]_\pm$	$\downarrow \sum_{m,\eta,\sigma,\tau,\zeta_{\pm(-)^m}} (-1)^m [s^r/\sigma\zeta_{\pm(-)^m})\cdot(((\lambda/Cm)\circ\eta)/\sigma D)]$
	$\times[(\zeta_{\pm(-)^m}/\tau)\cdot(\eta/D\tau)]$
$[\square;\lambda]''$	$\downarrow \sum_{\zeta,\eta,\rho,\sigma,\tau.\upsilon,\theta} (-1)^{\omega_\eta} [(((s^r/\zeta\sigma)\cdot(s^r/\eta\sigma))/\upsilon)$
	$\cdot(((\lambda/A)\circ\rho)/D\upsilon)] \times [(((\zeta'/\tau)\cdot(\eta'/\tau))/\theta)\cdot(\rho/D\theta)]$

Table II.3.1 continued

(d)	$SO(4rs+2r+2s+1) \downarrow SO(2r+1) \times SO(2s+1)$
$[1]$	$\downarrow [1] \times [1]$
Δ	$\downarrow \sum_{\zeta} [\Delta; s^r/\zeta] \times [\Delta; \zeta']$
$[\lambda]$	$\downarrow \sum_{\zeta} [((\lambda/C)\circ\eta)/D] \times [\eta/D]$
$[\Delta;\lambda]$	$\downarrow \sum_{\zeta,\eta,\sigma,\tau} [\Delta; (s^r/\zeta\sigma)\cdot(((\lambda/E)\circ\eta)/F\sigma)] \times [\Delta; (\zeta'/\tau)\cdot(\eta/F\tau)]$

Note : In this Table, m,n,r,s are positive integers;
$\zeta, \eta, \ldots$ are arbitrary S-functions of weight ω_ζ, $\omega_\eta \ldots$;
ζ' is the conjugate of ζ;
ζ_+ (ζ_-) is S-function of even (odd) weight;
A,B,E...are classical S-function series.

Table II.3.2 : Branching rules for basic spin irrep $\Lambda_\pm$ of SO_{4rs}, SO_{4rs+2s} for some special values of r and s

	r	s	Branching rules for $\Lambda_\pm$
(a)	1	n	$SO_{4n} \downarrow Sp_2 \times Sp_{2n}$ $\Lambda_\pm \downarrow \sum_{s_\pm}^{n} \langle n-s_\pm\rangle \times \langle 1^{s\pm}\rangle$
(b)	1	$n_+ + n_-$	$SO_{4(n_+ + n_-)} \downarrow (Sp_2 \times Sp_{n_+}) \times (Sp_2 \times Sp_{n_-})$ $\Lambda_+ \downarrow \sum_{s_\pm^+}^{n_+} \sum_{s_\pm^-}^{n_-} (\langle n_+ - s_+^+\rangle \times \langle 1^{s_+^+}\rangle \times \langle n_- - s_+^-\rangle \times \langle 1^{s_+^-}\rangle$ $+ \langle n_+ - s_-^+\rangle \times \langle 1^{s_-^+}\rangle \times \langle n_- - s_-^-\rangle \times \langle 1^{s_-^-}\rangle)$ $\Lambda_- \downarrow \sum_{s_\pm^-}^{n^-} \sum_{s_\pm^+}^{n^+} (\langle n_+ - s_+^+\rangle \times \langle 1^{s_+^+}\rangle \times \langle n_- - s_-^-\rangle \times \langle 1^{s_-^-}\rangle$ $+ \langle n_+ - s_-^+\rangle \times \langle 1^{s_-^+}\rangle \times \langle n_- - s_+^-\rangle \times \langle 1^{s_+^-}\rangle)$
(c)	2	n	$SO_{8n} \downarrow Sp_4 \times Sp_{2n}$ $\Lambda_\pm \downarrow \sum_{\zeta_\pm} \langle n^2/\zeta_\pm\rangle \times \langle \zeta'_\pm\rangle$
(d)	4	n	$SO_{16n+8} \downarrow SO_8 \times SO_{2n+1}$ $\Lambda_\pm \downarrow \sum_{\zeta} [\Lambda;\ \zeta]_{\pm(-)^{\omega_\zeta}} \times [4^n/\zeta']$
(e)	4	n	$SO_{16n} \downarrow SO_8 \times SO_{2n}$ $\Lambda_\pm \downarrow \sum_{\zeta_\pm(\zeta_1<n)} ([n^4/\zeta_\pm]_+ + [n^4/\zeta_\pm]_-) \times [\zeta'_\pm]$ $+ \sum_{\zeta_\pm,(\zeta_1=n)} [n^4/\zeta_\pm] \times ([\zeta'_\pm]_+ + [\zeta'_\pm]_-)$

Note: $s_+(s_-)$ is even (odd) integer, $\zeta_+(\zeta_-)$ is arbitrary partition of even (odd) weight ω_{ζ_+} (ω_{ζ_-}), ζ' is the conjugate of ζ.

irreps $\Lambda_{\pm}$ of SO_{2k} under the group-subgroup structure

$$SO_{2k} \supset SO_{D-2} \times K \qquad (I.4\text{-}1)$$

which is of special significance in determining the acceptable light-like representations of the extended D-dimensional Poincaré supersymmetry. The group K is the automorphism group (internal symmetry group), the nature of which depends upon the reality type of the basic spin irrep of SO_{D-2}: either SO(N), Sp(N) or SU(N) × U(1) for the real orthogonal, real symplectic (pseudoreal) or complex case respectively. SO_{D-2} is the space group or D-dimensional helicity group.

Various possible extended Poincaré supersymmetries together with their irreps have been considered by Strathdee (1987). The superalgebra is obtained from D-dimensional Poincaré algebra P by first extending it to include the generators of the automorphism group (internal symmetry algebra K) and then adjoining the set of fermionic supercharges $\{Q_j\}$ to the algebra. Thus the bosonic sector of the extended Poincaré supersymmetry is a direct sum of P and K while the fermionic sector is the Clifford algebra generated by supercharges Q_j, $j=1,2,\cdots,N$. The extended Poincaré superalgebra is sensitive to the space-time dimensionality D and the number of supercharges N. It was shown by Strathdee that if we require that the helicity of acceptable representations containing graviton to be not exceeding 2 in magnitude, then there is only limited possible values for D and N for light-like representations. The branching rules for light-like representations in these special cases have been given. (Note that a helicity 2 state corresponds to the irrep [2] of helicity group SO_{D-2}).

Using the results of previous sections, especially the group isomorphisms in table II.2.2 and the branching rules in table II.3.2, we are able to derive the branching rules listed in table (II.4.2) which holds for $D \leq 10$ and arbitrary N. For $D > 10$, a general method of calculating spin plethysms by using equivalent branching rules and the string method is described. This allows the light-like representations of extended Poincaré supersymmetry to be readily extended to higher helicity states and higher space-time dimensions if required. Explicit examples are given in table II.4.3. and II.4.5.

Following the notation of Strathdee, let $Q\ ^1/_2$ represent a set of N supercharges Q_j with 2k real components. This set generates the 2^k-dimensional complex Clifford algebra C_k which contains the important Lie subalgebras $C_k \supset SO_{2k+1} \supset SO_{2k}$. The 2^k-dimensional vector irrep of C_k denoted as $[1]_c$ corresponds to the spinor irrep Δ of SO_{2k+1} which reduces to two 2^{k-1}-dimensional irreps Δ_+, Δ_- of SO_{2k}

$$\begin{array}{ccccc} C_k & \supset & SO_{2k+1} & \supset & SO_{2k} \\ [1]_c & \downarrow & \Delta & \downarrow & \Delta_+ + \Delta_- \end{array} \qquad \text{(II.4-2)}$$

The irreps $\Delta_\pm$ of SO_{2k} may be further reduced by the action of subgroups of SO_{2k}. We need to fix the group-subgroup embedding $SO_{2k} \supset SO_{D-2} \times K$. This is achieved by considering the transformation properties of supercharges.

The 2k components of supercharges span the vector irreps of SO_{2k} and are required to be Lorentz spinors (for light-like case, basic spin irrep of SO_{D-2}). The N different supercharges Q_1, $Q_2, \cdots, Q_N$ must transform as vectors (span the vector irrep) of the automorphism group K Thus the nature of K should be matched to the reality type of basic spin irrep of SO_{D-2} which was given in table II.2.1. For example, for D=1,3 (mod 8) (or D-2 = 7,1 (mod 8)) character of basic spin irrep Δ is orthogonal hence K is the orthogonal group SO_N. In this case the decomposition of the vector irrep of SO_{2k} is given as

$$\begin{array}{llcll} D = 1,3 \pmod 8) : & SO_{2k} & \downarrow & SO_{D-2} \times SO_N & \\ & [1] & \downarrow & \Delta \times [1] & \text{(II.4-3a)} \\ & \text{with } 2k & = & 2^{(D-3)/2}\, N & \text{(II.4-3b)} \end{array}$$

Equation (II.4-3b) ensures that the dimensions are equal on both sides. Similarly, we obtain decompositions for [1] of SO_{2k} for other values of space-time dimension D. It is obvious that K depends on D and the pattern repeats with D (mod 8). Results are given in table (II.4.1).

The decomposition of the vector irrep [1] of SO_{2k} fixes the group-subgroup embedding and leaves the corresponding decompositions for the basic spin irrep $\Delta_\pm$ of SO_{2k} to be determined next.

For D=0,4 (mod 8) the required general results may be readily found by noting the equivalent group decompositions

$$\begin{tikzcd} SO_{2k} \arrow[rr] \arrow[dr] & & SO_{D-2} \times SU_N \times U_1 \\ & SU_k \times U_1 \arrow[ur] & \end{tikzcd}$$

The branching rule of $\Delta_\pm$ under $SO_{2k} \supset SU_k \times U_1$ is known (Black and Wybourne 1983):

$$\begin{array}{lcll} SO_{2k} & \downarrow & SU_k \times U_1 & \\ \Delta_\pm & \downarrow & \sum_{s_\pm}^{k} \{1^{s_\pm}\} \times \{s_\pm - k/2\} & \text{(II.4-4)} \end{array}$$

where the + or − sign is taken right through as appropriate and s_+ (s_-) are even (odd) integers. The branching rules of $\{1^{s_\pm}\}$ under $SU_k \supset SO_{D-2} \times U_N$ can be determined by first deciding the decomposition of {1} which fixes the embedding and then use the plethysm-branching rule theorem as mentioned in (II.2-4), leading to

Table II.4.1 : Decomposition of the vector irrep [1] of $SO_{2k} \downarrow SO_{D-2} \times K$

D=0,4(mod 8),	$2k=2^{\frac{D-2}{2}}N$	$SO_{2k} \downarrow SO_{D-2} \times SU_N \times U_1$ $[1] \downarrow \Delta_+ \times \{1\} \times \{1\} + \Delta_- \times \{\bar{1}\} \times \{\bar{1}\}$
D=1,3(mod 8)	$2k=2^{\frac{D-3}{2}}N$	$SO_{2k} \downarrow SO_{D-2} \times SO_N$ $[1] \downarrow \Delta \times [1]$
D=5,7(mod 8)	$2k=2^{\frac{D-3}{2}}N$	$SO_{2k} \downarrow SO_{D-2} \times Sp_N$ $[1] \downarrow \Delta \times \langle 1\rangle$
D=2(mod 8)	$2k=2^{\frac{D-4}{2}}(N_++N_-)$	$SO_{2k} \downarrow SO_{D-2} \times SO_{N_+} \times SO_{N_-}$ $[1] \downarrow \Delta_+ \times [1] \times [0] + \Delta_- \times [0] \times [1]$
	$2k=2^{\frac{D-4}{2}}N$	$SO_{2k} \downarrow SO_{D-2} \times SO_N$ $[1] \downarrow \Delta_+ \times [1]$
D=6(mod 8)	$2k=2^{\frac{D-4}{2}}(N_++N_-)$	$SO_{2k} \downarrow SO_{D-2} \times Sp_{N_+} \times Sp_{N_-}$ $[1] \downarrow \Delta_+ \times \langle 1\rangle \times \langle 0\rangle + \Delta_- \times \langle 0\rangle \times \langle 1\rangle$
	$2k=2^{\frac{D-4}{2}}N$	$SO_{2k} \downarrow SO_{D-2} \times Sp_N$ $[1] \downarrow \Delta_+ \times \langle 1\rangle$

For $D = 0,4 \pmod 8$, $2k = 2^{(D-2)/2}N$:

$$SU_k \downarrow SO_{D-2} \times SU_N$$
$$\{1\} \downarrow \Delta_+ \times \{1\}$$

thus

$$\{1^{s_\pm}\} \downarrow (\Delta_+ \times \{1\}) \otimes \{1^{s_\pm}\}$$
$$= \sum_{\zeta \vdash s_\pm} (\Delta_+ \otimes (\{1^{s_\pm}\}\circ\{\zeta\})) \times (\{1\} \otimes \{\zeta\})$$
$$= \sum_{\zeta \vdash s_\pm} (\Delta_+ \otimes \{\zeta\}) \times \{\zeta'\} \qquad \text{(II.4-5)}$$

where we have used the identities for plethysm and inner product (Wybourne 1970)

$$(A \times B) \otimes \{\lambda\} = \sum_{\zeta} A \otimes (\{\lambda\}\circ\{\zeta\}) \times B \otimes \{\zeta\} \qquad \text{(II.4-6)}$$
$$\{1^k\}\circ\{\lambda\} = \{\lambda'\} \quad \text{where} \quad \lambda \vdash k. \qquad \text{(II.4-7)}$$

Combining the two steps together finally yields

For $D = 0,4 \pmod 8$, $2k = 2^{(D-2)/2}N$

$$SO_{2k} \downarrow SO_{D-2} \times SU_N \times U_1$$
$$\Delta_\pm \downarrow \sum_{s_\pm}^{k} \sum_{\zeta \vdash s_\pm} [\Delta_+ \otimes \{\zeta\}] \times \{\zeta'\} \times \{s_\pm - k/2\}$$
$$\text{(II.4-8)}$$

Equation (II.4-8) simplifies for D=4 and 8. We have

For $D = 4$, $2k = 2N$:

$$SO_{2N} \downarrow SO_2 \times SU_N \times U_1$$
$$\Delta_\pm \downarrow \sum_{s_\pm}^{N} [s_\pm] \times \{1^{s_\pm}\} \times \{s_\pm - N/2\} \qquad \text{II.4-9)}$$

while for D = 8, due to the local isomorphism $SO_6 \sim SU_4$, $\Delta_+ \otimes \{\zeta\}$ can be calculated using the results of table (II.2-3), leading to

For D = 8, 2k = 8N:

$$\begin{aligned} SO_{8N} &\downarrow SO_6 \times SU_N \times U_1 \\ \Delta_\pm &\downarrow \sum_{s_\pm}^{4N} \sum_{\rho \vdash s_\pm} [\{\rho\}]' \times \{\rho'\} \times \{s_\pm - 2N\} \end{aligned} \qquad \text{(II.4-10)}$$

Using branching rules of table (II.3.2) together with the isomorphism of table (II.2.2) it was possible to obtain general results for D = 4 to 10 as given in table (II.4.2).

For D=5, (2k = 2N), $SO_{D-2} = SO_3 \sim Sp_2$, application of the branching rule (a) of table (II.3.2) and the isomorphism $Sp_2 \sim' SO_3$ $\langle a\rangle \sim' [a/2]$ (or $[\langle a\rangle]' = [a/2]$) readily yields

For D = 5, 2k = 2N:

$$\begin{aligned} SO_{2N} &\downarrow SO_3 \times Sp_N \\ \Delta_\pm &\downarrow \sum_{s_\pm}^{N} [\langle n-s_\pm\rangle]' \times \langle 1^{s_\pm}\rangle \\ &= \sum_{s_\pm}^{N} [(n-s_\pm)/2] \times \langle 1^{s_\pm}\rangle \end{aligned} \qquad \text{(II.4-11)}$$

Similarly, for D = 7, (2k = 4N), $SO_{D-2} = SO_5 \sim Sp_4$, by using rule (c) of table (II.3.2) and the isomorphism $\sim'$ between Sp_4 and SO_5 we get

For D = 7, 2k = 4N = 8n:

$$\begin{aligned} SO_{8n} &\downarrow SO_5 \times Sp_{2n} \\ \Delta_\pm &\downarrow \sum_{\zeta_\pm} [\langle n^2/\zeta_\pm\rangle]' \times \langle \zeta_\pm'\rangle \end{aligned} \qquad \text{(II.4.-12)}$$

where N = 2n, ζ' is conjugate of ζ. $n^2/\zeta_\pm$ is first evaluated in Sp_4 and

Table II.4.2 : Branching rules for the basic spin irreps of $SO_{2k} \downarrow SO_{D-2} \times K$ for $D \leq 10$

D=4	SO_{2N}	$\downarrow SO_2 \times SU_N \times U_1$	
	$\Delta_\pm$	$\downarrow \sum_{s_\pm}^{N} [s_\pm]_+ \times \{1^{s_\pm}\} \times \{s_\pm - N/2\}$	
D=5	SO_{4n}	$\downarrow SO_3 \times Sp_{2n}$	N=2n
	$\Delta_\pm$	$\downarrow \sum_{s_\pm}^{n} [(n-s_\pm)/2] \times \langle 1^{s_\pm} \rangle$	
D=6	SO_{4n}	$\downarrow SO_4 \times Sp_{2n}$	
	$\Delta_\pm$	$\downarrow \sum_{s_\pm}^{n} [(n-s_\pm)/2, (n-s_\pm)/2] \times \langle 1^{s_\pm} \rangle$	
	$SO_{4(n_+ + n_-)}$	$\downarrow SO_4 \times Sp_{2n_+} \times Sp_{2n_-}$	$N=2(n_+ + n_-)$
	Δ_+	$\downarrow \sum_{s^+_\pm, s^-_\pm}^{n_+ n_-} [(n_+ + n_- - s^+_\pm - s^-_\pm)/2, (n_+ - n_- - s^+_\pm + s^-_\pm)/2] \times \langle 1^{s^+_\pm} \rangle \times \langle 1^{s^-_\pm} \rangle$	
	Δ_-	$\downarrow \sum_{s^+_\pm, s^-_\mp}^{n_+ n_-} [(n_+ + n_- - s^+_\pm - s^-_\mp)/2, (n_+ - n_- - s^+_\pm + s^-_\mp)/2] \times \langle 1^{s^+_\pm} \rangle \times \langle 1^{s^-_\mp} \rangle$	
D=7	SO_{8n}	$\downarrow SO_5 \times Sp_{2n}$	N=2n
	$\Delta_\pm$	$\downarrow \sum_{\zeta_\pm} [\langle n^2/\zeta_\pm \rangle]' \times \langle \zeta'_\pm \rangle$	
D=8	SO_{8N}	$\downarrow SO_6 \times SU_N \times U_1$	
	$\Delta_\pm$	$\downarrow \sum_{s_\pm}^{4N} \sum_{\zeta \vdash s_\pm} [\{\zeta\}]' \times \{\zeta'\} \times \{s_\pm - 2N\}$	
D=10	SO_{16n+8}	$\downarrow SO_8 \times SO_{2n+1}$	N=2n+1
	$\Delta_\pm$	$\downarrow \sum_{\zeta} [[\Delta;\zeta]_{\pm(-)^{\omega_\zeta}}]' \times [4^n/\zeta']$	
	SO_{16n}	$\downarrow SO_8 \times SO_{2n}$	N=2n
	$\Delta_\pm$	$\downarrow \sum_{\zeta_\pm} ([[n^4/\zeta_\pm]_+]' + [[n^4/\zeta_\pm]_-]') \times [\zeta'_\pm]$	$(\zeta_1 < n)$
		$+ \sum_{\zeta_\pm} [[n^4/\zeta_\pm]]' \times ([\zeta'_\pm]_+ + [\zeta'_\pm]_-)$	$(\zeta_1 = n)$

Note: For D=9, branching rule for the basic spin irrep $\Delta_\pm$ of $SO_{16n+8} \downarrow SO_7 \times SO_{2n+1}$ and $SO_{16n} \downarrow SO_7 \times SO_{2n}$ can be obtained from the D=10 result by using branching rule $SO_8 \downarrow SO_7$, $[\lambda] \downarrow [\lambda]/M$ given in equation (II.2-21). For D=6 the summation $\Sigma_{s^+_\pm s^-_\pm}$ is to be understood as first being made over (s^+_+, s^-_+) and then over (s^+_-, s^-_-) while for $\Sigma_{s^+_\pm s^-_\mp}$, the sum is first over (s^+_+, s^-_-) and then over (s^+_-, s^-_+) where $s^+_\pm \leq n_+$, $s^-_\pm \leq n_-$.

transcribed into SO_5 irrep labels by using the isomorphism of table (II.2.2) i.e.

$$Sp_4 \sim' SO_5 : [\langle a,b \rangle]' = [(a+b)/2 \ (a-b)/2]. \qquad \text{(II.4-13)}$$

For D = 10 (2k = 8N), the group-subgroup combination and the embedding is (Table II.4.1)

$$\begin{array}{l} SO_{8N} \downarrow SO_8 \times SO_N \\ [1] \downarrow \Delta_+ \times [1] \end{array} \qquad \text{(II.4-14)}$$

We have already obtained the decomposition of $\Delta_\pm$ for the same group-subgroup structure, but for different embedding (Table II.3.2):

$$\begin{array}{l} SO_{8N} \downarrow SO_8 \times SO_N \\ [1] \downarrow [1] \times [1] \end{array} \qquad \text{(II.4-15)}$$

Under the automorphism of SO_8, $[1] \sim' \Delta_+$ or $[[1]]' = \Delta_+$. So we need only apply this automorphism $\sim'$ to the branching rules formula (d), (e) of table II.3.2, corresponding to the embedding (II.4-14) in order to achieve the desired decompositions.

For D = 9 (2k = 8N), the required branching is $SO_{8N} \downarrow SO_7 \times SO_N$ with embedding $[1] \downarrow \Delta \times [1]$. Consider the equivalent structure

$$\begin{array}{l} SO_{8N} \downarrow SO_8 \times SO_N \downarrow SO_7 \times SO_N \\ [1] \downarrow \Delta_+ \times [1] \downarrow \Delta \times [1] \end{array} \qquad \text{(II.4-16)}$$

Thus by making use of the result for D = 10 and the branching rule for $SO_8 \downarrow SO_7$ as given in equation (II.2-21), the required branching rule

Table II.4.3 : Branching rules for the basic spin irreps of $SO_{2k} \downarrow SO_{D-2} \times K$ for D=5 to 10, explicit results.

D=5

N=2 $SO_4 \downarrow SO_3 \times Sp_2$

$\Delta_+ \downarrow \Delta\times\langle 0\rangle$

$\Delta_- \downarrow [0]\times\langle 1\rangle$

N=4 $SO_8 \downarrow SO_3 \times Sp_4$

$\Delta_+ \downarrow [1]\times\langle 0\rangle+[1]\times\langle 1^2\rangle$

$\Delta_- \downarrow \Delta\times\langle 1\rangle$

N=6 $SO_{12} \downarrow SO_3 \times Sp_6$

$\Delta_+ \downarrow [\Delta;1]\times\langle 0\rangle+\Delta\times\langle 1^2\rangle$

$\Delta_- \downarrow [1]\times\langle 1\rangle+[0]\langle 1^3\rangle$

N=8 $SO_{16} \downarrow SO_3 \times Sp_8$

$\Delta_+ \downarrow [2]\times\langle 0\rangle+[1]\times\langle 1^2\rangle+[0]\times\langle 1^4\rangle$

$\Delta_- \downarrow [\Delta;1]\times\langle 1\rangle + \Delta\times\langle 1^3\rangle$

N=10 $SO_{20} \downarrow SO_3 \times Sp_{10}$

$\Delta_+ \downarrow [\Delta;2]\times\langle 0\rangle+[\Delta;1]\times\langle 1^2\rangle+\Delta\times\langle 1^4\rangle$

$\Delta_- \downarrow [2]\times\langle 1\rangle+[1]\times\langle 1^3\rangle+[0]\times\langle 1^5\rangle$

N=12 $SO_{24} \downarrow SO_3 \times Sp_{12}$

$\Delta_+ \downarrow [3]\times\langle 0\rangle+[2]\times\langle 1^2\rangle+[1]\times\langle 1^4\rangle+[0]\times\langle 1^6\rangle$

$\Delta_- \downarrow [\Delta;2]\times\langle 1\rangle+[\Delta;1]\times\langle 1^3\rangle+\Delta\times\langle 1^5\rangle$

Table II.4.3 continued on next page

D=6

N=2 $SO_4 \downarrow SO_4 \times Sp_2$

$\Delta_+ \downarrow \Delta_+\times\langle 0\rangle$

$\Delta_- \downarrow [0]\times\langle 1\rangle$

N=4 $SO_8 \downarrow SO_4 \times Sp_4$

$\Delta_+ \downarrow [1^2]_+\times\langle 0\rangle+[0]\times\langle 1^2\rangle$

$\Delta_- \downarrow \Delta_+\times\langle 1\rangle$

$SO_8 \downarrow SO_4 \times Sp_2 \times Sp_2$

$\Delta_+ \downarrow [1]\times\langle 0\rangle\times\langle 0\rangle+[0]\times\langle 1\rangle\times\langle 1\rangle$

$\Delta_- \downarrow \Delta_+\times\langle 0\rangle\times\langle 1\rangle+\Delta_-\times\langle 1\rangle\times\langle 0\rangle$

N=6 $SO_{12} \downarrow SO_4 \times Sp_6$

$\Delta_+ \downarrow [\Delta;1^2]_+\times\langle 0\rangle+\Delta_+\times\langle 1^2\rangle$

$\Delta_- \downarrow [1^2]_+\times\langle 1\rangle+[0]\times\langle 1^3\rangle$

$SO_{12} \downarrow SO_4 \times Sp_4 \times Sp_2$

$\Delta_+ \downarrow [\Delta;1]_+\times\langle 0\rangle\times\langle 0\rangle+\Delta_+\times\langle 1\rangle\times\langle 1\rangle+\Delta_-\times\langle 1^2\rangle\times\langle 0\rangle$

$\Delta_- \downarrow [1^2]_+\times\langle 0\rangle\times\langle 1\rangle+[0]\times\langle 1^2\rangle\times\langle 1\rangle+[1]\times\langle 1\rangle\times\langle 0\rangle$

N=8 $SO_{16} \downarrow SO_4 \times Sp_8$

$\Delta_+ \downarrow [2^2]_+\times\langle 0\rangle+[1^2]_+\times\langle 1^2\rangle+[0]\times\langle 1^4\rangle$

$\Delta_- \downarrow [\Delta;1^2]_+\times\langle 1\rangle+\Delta_+\times\langle 1^3\rangle$

$SO_{16} \downarrow SO_4 \times Sp_6 \times Sp_2$

$\Delta_+ \downarrow [21]_+\times\langle 0\rangle\times\langle 0\rangle+[1]\times\langle 1^2\rangle\times\langle 0\rangle+[1^2]_+\times\langle 1\rangle\times\langle 1\rangle+[0]\times\langle 1^3\rangle\times\langle 1\rangle$

$\Delta_- \downarrow [\Delta;1^2]_+\times\langle 0\rangle\times\langle 1\rangle+\Delta_+\times\langle 1^2\rangle\times\langle 1\rangle+[\Delta;1]_+\times\langle 1\rangle\times\langle 0\rangle+\Delta_-\times\langle 1^3\rangle\times\langle 0\rangle$

Table II.4.3 continued on next page

$SO_{16} \downarrow SO_4 \times Sp_4 \times Sp_4$

$\Delta_+ \downarrow [2]\times\langle 0\rangle\times 0\rangle+[1^2]_-\times\langle 1^2\rangle\times\langle 0\rangle+[1^2]_+\times\langle 0\rangle\times\langle 1^2\rangle$

$+[0]\times\langle 1^2\rangle\times\langle 1^2\rangle$

$\Delta_- \downarrow [\Delta;1]_+\times\langle 0\rangle\times\langle 1\rangle+[\Delta;1]_-\times\langle 1\rangle\times\langle 0\rangle+\Delta_-\times\langle 1^2\rangle\times\langle 1\rangle+\Delta_+\times\langle 1\rangle\times\langle 1^2\rangle$

D=7

N=2 $SO_8 \downarrow SO_5 \times Sp_2$

$\Delta_+ \downarrow [1]\times\langle 0\rangle+[0]\times\langle 2\rangle$

$\Delta_- \downarrow \Delta\times\langle 1\rangle$

N=4 $SO_{16} \downarrow SO_5 \times Sp_4$

$\Delta_+ \downarrow [2]\times\langle 0\rangle+[1^2]\times\langle 1^2\rangle+[1]\times\langle 2\rangle$

$\Delta_- \downarrow [\Delta;1]\times\langle 1\rangle+\Delta\times\langle 21\rangle$

N=6 $SO_{24} \downarrow SO_5 \times Sp_6$

$\Delta_+ \downarrow [3]\times\langle 0\rangle+[21]\times\langle 1^2\rangle+[2]\times\langle 2\rangle+[1^2]\times\langle 21^2\rangle+[1]\times\langle 2^2\rangle+[0]\times\langle 2^3\rangle$

$\Delta_- \downarrow [\Delta;2]\times\langle 1\rangle+[\Delta;1^2]\times\langle 1^3\rangle+[\Delta;1]\times\langle 21\rangle+\Delta\times\langle 2^21\rangle$

N=8 $SO_{32} \downarrow SO_5 \times Sp_8$

$\Delta_+ \downarrow [4]\times\langle 0\rangle+[31]\times\langle 1^2\rangle+[3]\times\langle 2\rangle+[2^2]\times\langle 1^4\rangle$

$+[21]\times\langle 21^2\rangle+[2]\times\langle 2^2\rangle+[1^2]\times\langle 2^21^2\rangle+[1]\times\langle 2^3\rangle+[0]\times\langle 2^4\rangle$

$\Delta_- \downarrow [\Delta;3]\times\langle 1\rangle+[\Delta;21]\times\langle 3\rangle+[\Delta;2]\times\langle 21\rangle+[\Delta;1^2]\times\langle 21^3\rangle$

$+[\Delta;1]\times\langle 2^21\rangle+\Delta\times\langle 2^31\rangle$

Table II.4.3 continued on next page

D=8

N=1 $SO_8 \downarrow SO_6 \times U_1$

$\Delta_+ \downarrow [0]\times\{\bar{2}\}+[1]\times\{0\}+[0]\times\{2\}$

$\Delta_- \downarrow \Delta_+\times\{\bar{1}\}+\Delta_-\times\{1\}$

N=2 $SO_{16} \downarrow SO_6 \times SU_2 \times U_1$

$\Delta_+ \downarrow [0]\times\{0\}\times(\{\bar{4}\}+\{4\})+[1^3]_+\times\{0\}\times\{\bar{2}\}+[1^3]_-\times\{0\}\times\{2\}$
$+[1]\times\{2\}\times(\{\bar{2}\}+\{2\})+([0]\times\{4\}+[2]\times\{0\}+[1^2]\times\{2\})\times\{0\}$

$\Delta_- \downarrow \Delta_+\times\{1\}\times\{\bar{3}\}+\Delta_-\times\{1\}\times\{3\}+\Delta_+\times\{3\}\times\{1\}+\Delta_-\times\{3\}\times\{\bar{1}\}$
$+[\Delta;1]_+\times\{1\}\times\{\bar{1}\}+[\Delta;1]_-\times\{1\}\times\{1\}$

N=3 $SO_{24} \downarrow SO_6 \times SU_3 \times U_1$

$\Delta_+ \downarrow [0]\times\{0\}\times(\{\bar{6}\}+\{6\})+([1]\times\{2\}+[1^3]_+\times\{1^2\})\times\{\bar{4}\}+([1]\times\{2^2\}$
$+[1^3]_-\times\{1\})\times\{4\}+([21^2]_+\times\{1\}+[2]\times\{2^2\}+[1^2]\times\{31\})\times\{\bar{2}\}$
$+([21^2]_-\times\{1^2\}+[2]\times\{2\}+[1^2]\times\{32\})\times\{2\}$
$+([3]\times\{0\}+[1^3]_+\times\{3\}+[1^3]_-\times\{3^2\}+[1]\times\{42\}+[21]\times\{21\})\times\{0\}$

$\Delta_- \downarrow \Delta_-\times\{1\}\times\{\bar{5}\}+\Delta_+\times\{1^2\}\times\{5\}+([\Delta;1^3]_+\times\{0\}+\Delta_-\times\{3\}$
$+[\Delta;1]_+\times\{21\})\times\{3\}+([\Delta;1^3]_-\times\{0\}+\Delta_+\times\{3^2\}+[\Delta;1]_-\times\{21\})\times\{\bar{3}\}$
$+(\Delta_-\times\{41\}+[\Delta;1]_+\times\{1^2\}+[\Delta;1]_-\times\{32\}+[\Delta;1^2]_+\times\{21\})\times\{\bar{1}\}$
$+(\Delta_+\times\{43\}+[\Delta;1]_-\times\{1\}+[\Delta;1]_+\times\{31\}+[\Delta;1^2]_-\times\{21\})\times\{1\}$

Table II.4.3 continued on next page

D=9

N=1 $SO_8 \downarrow SO_7 \times SO_1$

$\Delta_+ \downarrow \Delta$

$\Delta_- \downarrow [1]+[0]$

N=2 $SO_{16} \downarrow SO_7 \times SO_2$

$\Delta_+ \downarrow ([1^3]+[2]+[1]+[0])\times[0]+([1^2]+[1])\times([2]_++[2]_-)$
$+[0]\times([4]_++[4]_-)$

$\Delta_- \downarrow ([\Delta;1]+\Delta)\times([1]_++[1]_-)+\Delta\times([3]_++[3]_-)$

N=3 $SO_{24} \downarrow SO_7 \times SO_3$

$\Delta_+ \downarrow ([1]+[0])\times[4]+([\Delta;1]+\Delta)\times[3]+([\Delta;1^2]+[\Delta;1])\times[2]$
$+([\Delta;2]+[\Delta;1]+\Delta)\times[1]+[\Delta;1^3]\times[0]$

$\Delta_- \downarrow ([1]+[0])\times[4]+([1^3]+[1^2])\times[3]+([21]+[2]+[1^2]$
$+[1])\times[2]+([21^2]+[1^3])\times[1]+([3]+[2]+[1]+[0])\times[0]$

N=4 $SO_{32} \downarrow SO_7 \times SO_4$

$\Delta_+ \downarrow ([2^3]+[4]+[3]+[2]+[1]+[0])\times[0]+([31^2]+[21^2]$
$+[1^3])\times([1^2]_++[1^2]_-)+([2^21]+[21^2]+[31]+[3]+[21]$
$+[2]+[1^2]+[1])\times[2]+([2^2]+[21]+[2])\times([2^2]_++[2^2]_-)$
$+([21^2]+[21]+[1^3]+[1^2])\times([31]_++[31]_-)+([1^3]$
$+[2]+[1]+[0])\times[4]+[1^3]\times([3^2]_++[3^2]_-)+([1^2]$
$+[1])\times([42]_++[42]_-)+[0]\times([4^2]_++[4^2]_-)$

$\Delta_- \downarrow ([\Delta;21^2]+[\Delta;1^3]+[\Delta;3]+[\Delta;2]+[\Delta;1]+\Delta)\times[1]+([\Delta;21]$
$+[\Delta;2]+[\Delta;1^2]+[\Delta;1])\times([21]_++[21]_-)+([\Delta;1^3]$
$+[\Delta;1^2]+[\Delta;2]+[\Delta;1]+\Delta)\times[3]+([\Delta;1^2]+[\Delta;1])\times([32]_+$
$+[32]_-)+([\Delta;1]+\Delta)\times([41]_++[41]_-)+\Delta\times([43]_++[43]_-)$

Table II.4.3 continued on next page

D=10

N=1 $SO_8 \downarrow SO_8$

$\Delta_+ \downarrow \Delta_-$

$\Delta_- \downarrow [1]$

N=2 $SO_{16} \downarrow SO_8 \times SO_2$

$\Delta_+ \downarrow ([1^4]_-+[2])\times[0]+[1^2]\times([2]_++[2]_-)+[0]\times([4]_++[4]_-)$

$\Delta_- \downarrow [\Delta;1]_-\times([1]_++[1]_-)+\Delta_+\times([3]_++[3]_-)$

N=3 $SO_{24} \downarrow SO_8 \times SO_3$

$\Delta_+ \downarrow \Delta_-\times[4]+[\Delta;1]_+\times[3]+[\Delta;1^2]_-\times[2]+[\Delta;2]_-\times[1]+[\Delta;1^4]_-\times[0]$

$\Delta_- \downarrow [1]\times[4]+[1^3]\times[3]+[21]\times[2]+[21^3]_-\times[1]+[3]\times[0]$

N=4 $SO_{32} \downarrow SO_8 \times SO_4$

$\Delta_+ \downarrow ([2^4]_-+[4])\times[0]+[31^3]_-\times([1^2]_++[1^2]_-)+([2^21^2]_-+[31])\times[2]$
$+[2^2]\times([2^2]_++[2^2]_-)+[21^2]\times([31]_++[31]_-)+([1^4]_-+[2])\times[4]$
$+[1^4]_+\times([3^2]_++[3^2]_-)+[1^2]\times([42]_++[42]_-)$
$+[0]\times([4^2]_++[4^2]_-)$

$\Delta_- \downarrow [\Delta;21^3]_-\times[1]+[\Delta;3]_-\times[1]+[\Delta;21]_-\times([21]_++[21]_-)$
$+([\Delta;1^3]_-+[\Delta;2]_+)\times[3]+[\Delta;1^2]_+\times([32]_++[32]_-)$
$+[\Delta;1]_-\times([41]_++[41]_-)+\Delta_+\times([43]_++[43]_-)$

N=5 $SO_{40} \downarrow SO_8 \times SO_5$

$\Delta_+ \downarrow \Delta_+\times[4^2]+[\Delta;1]_+\times[43]+[\Delta;1^2]_-\times[42]+[\Delta;1^3]_+\times[3^2]$
$+[\Delta;2]_-\times[41]+[\Delta;21]_+\times[32]+[\Delta;1^4]_-\times[4]+[\Delta;21^2]_-\times[31]$
$+[\Delta;2^2]_-\times[2^2]+[\Delta;3]_+\times[3]+[\Delta;31]_-\times[21]+[\Delta;2^21^2]_-\times[2]$
$+[\Delta;31^3]_-\times[1^2]+[\Delta;4]_-\times[1]+[\Delta;2^4]_-\times[0]$

$\Delta_- \downarrow [1]\times[4^2]+[1^3]\times[43]+[21]\times[42]+[21^3]_-\times[3^2]$
$+[21^3]_-\times[41]+[2^21]\times[32]+[3]\times[4]+[31^2]\times[31]$
$+[32]\times[2^2]+[2^31]_-\times[3]+[321^2]_-\times[21]+[41]\times[2]$
$+[41^3]_-\times[1^2]+[32^3]_-\times[1]+[5]\times[0]$

follows immediately.

Some explicit decomposition of $\Delta_\pm$ under $SO_{2k} \downarrow SO_{D-2} \times K$ for $D \lesssim 10$ and special values of N using these branching rules have been listed in table II.4.3.

The results in table II.4.2 exhaust the possibilities for exploiting isomorphism. For D = 0, 4 (mod 8) equation (II.4-8) gives the general results. For other values of D ($D > 10$, $D \neq 0,4$ (mod 8)) it is possible to make use of the group-subgroup structure:

$$\begin{tikzcd}
 & SO_{D-2} \times K \arrow[dr] & \\
SO_{2k} \arrow[ur] \arrow[dr] & & SO_{D-4} \times U_1 \times K \\
 & SU_k \times U_1 \arrow[ur] &
\end{tikzcd}$$

The lower chain can be evaluated using (II.4-4) and the properties of plethysm (II.2-4) (Littlewood 1950, Wybourne 1970) to yield the results given in table II.4.4. As an example, we derive the first rule in table II.4.4.

For D = 1,3(mod 8), $2k = 2^{(D-3)/2}N$:

$$SO_{2k} \downarrow SU_k \times U$$

$$\Delta_\pm \downarrow \sum_{s_\pm}^{k} \{1^{s_\pm}\} \times \{s_\pm - k/2\} \qquad \text{by (II.4-4)}$$

$$SU_k \downarrow SO_{D-4} \times SO_N$$

$$\{1\} \downarrow \Delta \times [1]$$

$$\{1^{s_\pm}\} \downarrow (\Delta \times [1]) \otimes \{1^{s_\pm}\} \qquad \text{by (II.2-4)}$$

$$= \sum_{\rho \vdash s_\pm} [\Delta \otimes \{\rho\}] \times ([1] \otimes \{\rho'\}) \qquad \text{by (II.4-6)}$$

$$= \sum_{\rho \vdash s_\pm} [\Delta \otimes \{\rho\}] \times [\rho'/D] \qquad \text{by (II.2-16b)}$$

Table II.4.4 : Decomposition of $\Delta_\pm$ under $SO_{2k} \downarrow SO_{D-4} \times U_1 \times K$

D=1,3(mod 8) $\qquad D \geq 9,\ 2k=2^{\frac{D-3}{2}}N$

$$SO_{2k} \downarrow SO_{D-4} \times U_1 \times SO_N$$
$$\Delta_\pm \downarrow \sum_{s_\pm} \sum_{\rho\vdash s_\pm} [\Delta \otimes \{\rho\}]\times\{s_\pm - k/2\} \times [\rho'/D]$$

D=5,7(mod 8) $\qquad D \geq 7,\ 2k=2^{\frac{D-3}{2}}N$

$$SO_{2k} \downarrow SO_{D-4} \times U_1 \times Sp_N$$
$$\Delta_\pm \downarrow \sum_{s_\pm} \sum_{\rho\vdash s_\pm} [\Delta\otimes\{\rho\}]\times\{s_\pm - k/2\}\times\langle\rho'/B\rangle$$

D=2(mod 8) $\qquad D \geq 10,\ 2k=2^{\frac{D-4}{2}}(N_+ + N_-)$

$$SO_{2k} \downarrow SO_{D-4} \times U_1 \times SO_{N_+} \times SO_{N_-}$$
$$\Delta_\pm \downarrow \sum_{s_\pm} \sum_{\rho\vdash s_\pm} \sum_{\eta\vdash k-s_\pm} [(\Delta_+\otimes\{\rho\})\cdot(\Delta_-\otimes\{\eta\})]\times\{s_\pm - k/2\}\times[\rho'/D] \times [\eta'/D]$$
$$S_{2k} \downarrow SO_{D-4} \times U_1\ SO_N$$
$$\Delta_\pm \downarrow \sum_{s_\pm} \sum_{\rho\vdash s_\pm} [\Delta_+\otimes\{\rho\}]\times\{s_\pm - k/2\}\times[\rho'/D]$$

D=6(mod 8) $\qquad D \geq 14,\ 2k=2^{\frac{D-4}{2}}(N_+ + N_-)$

$$SO_{2k} \downarrow SO_{D-4} \times U_1 \times Sp_{N_+} \times Sp_{N_-}$$
$$\Delta_\pm \downarrow \sum_{s_\pm} \sum_{\rho\vdash s_\pm} \sum_{\eta\vdash k-s_\pm} [(\Delta_+\otimes\{\rho\})\cdot(\Delta_-\otimes\{\eta\})]\times\{s_\pm - k/2\}\times\langle\rho'/B\rangle\times\langle\eta'/B\rangle$$
$$SO_{2k} \downarrow SO_{D-4} \times U_1 \times Sp_n$$
$$\Delta_\pm \downarrow \sum_{s_\pm} \sum_{\rho\vdash s_\pm} [\Delta_+\otimes\{\rho\}]\times\{s_\pm - k/2\}\times\langle\rho'/B\rangle$$

Combine the two steps to yield

$$\begin{array}{ll} SO_{2k} & \downarrow SO_{D-4} \times U_1 \times SO_N \\ \Delta_\pm & \downarrow \sum\limits_{s_\pm}^{k} \sum\limits_{\rho\vdash s_\pm} [\Delta \otimes \{\rho\}] \times \{s_\pm - k/2\} \times [\rho'/D] \end{array} \tag{II.4-17}$$

Other results in table II.4.4 were obtained in a similar manner.

Having obtained the results for the lower chain, the irreps of $SO_{D-2} \times K$ that recover those of $SO_{D-4} \times U_1 \times K$ can be found by the string method (Wybourne 1984). Explicit example decomposition of $\Delta_\pm$ for $D = 11$ and $N = 1,2,3$ have been given in table II.4-5.

The principal difficulty in implementing the results is the evaluation of the relevant spin plethysms of SO_{D-4}. Symmetrized Kronecker powers higher than four of basic spin irreps of SO_n for $n > 11$ are not generally known to this point. Table II.2.3-5 only allows $D < 14$.

Table II.4.5 : Branching rules for the basic spin irreps under $SO_{16N} \downarrow SO_N \times SO_9$ for N=1,2,3, explicit results.

D=11

N=1 $\quad SO_{16} \downarrow SO_9$

$\Delta_+ \downarrow [2]+[1^3]$

$\Delta_- \downarrow [\Delta;1]$

N=2 $\quad SO_{32} \downarrow O_2 \times SO_9$

$\Delta_+ \downarrow [0] \times ([4]+[31^3]+[31^2]+[3]+[2^3]+[2^2 1]+[2^2]+[21^3]$

$+[21^2]+[2]+[1^4]+[1^3]+[1]+[0])$

$+[2] \times ([31^2]+[31]+[2^2 1^2]+[21^2]+[21]+[1^3]+[1^2])$

$+[4] \times ([2^2]+[21^3]+[21]+2]+[1^4])$

$+[6] \times ([1^3]+[1^2])$

$+[8] \times [0]$

$\Delta_- \downarrow [1] \times ([\Delta;3]+[\Delta;21^2]+[\Delta;21]+[\Delta;2]+[\Delta;1^3]+[\Delta;1^2]$

$+[\Delta;1]+\Delta)$

$+[3] \times ([\Delta;21]+[\Delta;2]+[\Delta;1^4]+[\Delta;1^2]+[\Delta;1])$

$+[5] \times ([\Delta;1^2]+[\Delta;1])$

$+[7] \times \Delta$

N=3 $\quad SO_{48} \downarrow O_3 \times SO_9$

$\Delta_+ \downarrow [0] \times ([6]+[51^2]+[42^2]+[42]+[41^3]+[4]+[3^3]+[3^2 1]+[32^2 1]$

$+[321^2]+2[31^2]+[31]+2[2^3]+[2^2 1^2]+[2^2]+[21^3]+[2]$

$+2[1^3]+[0])$

$+[1] \times ([51^3]+[5]+[42^2 1]+[421^2]+[421]+[41^3]+2[41^2]$

$+[41]+[3^2 2]+[3^2 1^2]+[32^2 1]+2[32^2]+2[321^2]$

$+2[321]+[32]+3[31^3]+[31^2]+[31]+[3]+2[2^3 1]$

$+2[2^2 1^2]+2[2^2 1]+2[21^3]+3[21^2]+2[21]+2[1^4]$

$+[1^2]+[1])$

$+[2] \times ([51^2]+[51]+[42^2]+[421^2]+[421]+[42]+2[41^3]$

$+[41^2]+[41]+[4]+[3^2 21]+[3^2 1]+2[32^2 1]+[32^2]$

$+3[321^2]+3[321]+[32]+2[31^3]+4[31^2]+2[31]+[2^4]$

Table II.4.5 continued on next page

Table II.4.5 continued

$$
\begin{aligned}
&\quad +[2^31]+2[2^3]+3[2^21^2]+2[2^21]+2[2^2]+4[21^3]\\
&\quad +2[21^2]+2[21]+2[2]+[1^4]+2[1^3]+[1^2])\\
+[3] \times &\ ([421^2]+[421]+[41^3]+2[41^2]+[41]+[3^21^2]+[3^2]\\
&\quad +[32^3]+[32^21]+[32^2]+3[321^2]+2[321]+2[32]\\
&\quad +4[31^3]+2[31^2]+2[31]+2[3]+2[2^31]+[2^3]+3[2^21^2]\\
&\quad +3[2^21]+[2^2]+3[21^3]+4[21^2]+2[21]+2[1^4]+[1^3]\\
&\quad +[1^2]+[1])\\
+[4] \times &\ ([42]+[41^3]+[41]+[4]+[3^21]+[32^21]+2[321^2]\\
&\quad +2[321]+[32]+2[31^3]+3[31^2]+2[31]+[2^4]+[2^31]\\
&\quad +[2^3]+3[2^21^2]+2[2^21]+2[2^2]+4[21^3]+2[21^2]\\
&\quad +2[21]+2[2]+[1^4]+2[1^3]+[1^2])\\
+[5] \times &\ ([321^2]+[321]+[32]+2[31^3]+[31^2]+[31]+[3]\\
&\quad +[2^31]+2[2^21^2]+2[2^21]+[2^2]+2[21^3]+3[21^2]\\
&\quad +2[21]+2[1^4]+[1^3]+[1^2]+[1])\\
+[6] \times &\ ([31^2]+[31]+[2^3]+[2^21^2]+[2^21]+[2^2]+2[21^3]+[21^2]\\
&\quad +[21]+[2]+[1^4]+2[1^3]+[1^2]+[0])\\
+[7] \times &\ ([21^3]+[21^2]+[21]+[1^4]+[1^2]+[1])\\
+[8] \times &\ ([2]+[1^3])
\end{aligned}
$$

$$
\begin{aligned}
\Delta_- \downarrow [0] \times &\ ([\Delta;41^3]+[\Delta;4]+[\Delta;321]+[\Delta;31^2]+[\Delta;31]+[\Delta;2^21]\\
&\quad +[\Delta;21^3]+[\Delta;21^2]+[\Delta;21]+[\Delta;2]+[\Delta;1^4]+[\Delta;1^2]+[\Delta;1])\\
+[1] \times &\ ([\Delta;5]+[\Delta;41^2]+[\Delta;41]+[\Delta;4]+[\Delta;32^2]+[\Delta;321]\\
&\quad +[\Delta;32]+[\Delta;31^3]+2[\Delta;31^2]+2[\Delta;31]+2[\Delta;3]+[\Delta;2^3]\\
&\quad +[\Delta;2^21^2]+2[\Delta;2^21]+[\Delta;2^2]+[\Delta;21^3]+3[\Delta;21^2]+3[\Delta;21]\\
&\quad +2[\Delta;2]+[\Delta;1^4]+2[\Delta;1^3]+2[\Delta;1^2]+2[\Delta;1]+\Delta)\\
+[2] \times &\ ([\Delta;41^2]+[\Delta;41]+[\Delta;4]+[\Delta;321^2]+[\Delta;321]+[\Delta;32]\\
&\quad +[\Delta;31^3]+3[\Delta;31^2]+3[\Delta;31]+2[\Delta;3]+[\Delta;2^3]+[\Delta;2^21^2]\\
&\quad +2[\Delta;2^21]+2[\Delta;2^2]+2[\Delta;21^3]+4[\Delta;21^2]+4[\Delta;21]\\
&\quad +3[\Delta;2]+[\Delta;1^4]+3[\Delta;1^3]+3[\Delta;1^2]+2[\Delta;1]+\Delta)
\end{aligned}
$$

Table II.4.5 continued on next page

Table II.4.5 continued

$$
\begin{aligned}
&+[3]\times([\Delta;41]+[\Delta;4]+[\Delta;321]+[\Delta;32]+[\Delta;31^3]+2[\Delta;31^2]\\
&\qquad+3[\Delta;31]+2[\Delta;3]+[\Delta;2^31]+[\Delta;2^21^2]+2[\Delta;2^21]\\
&\qquad+2[\Delta;2^2]+2[\Delta;21^3]+4[\Delta;21^2]+5[\Delta;21]+3[\Delta;2]+2[\Delta;1^4]\\
&\qquad+2[\Delta;1^3]+3[\Delta;1^2]+3[\Delta;1])\\
&+[4]\times([\Delta;32]+[\Delta;31^3]+[\Delta;31^2]+2[\Delta;31]+2[\Delta;3]+[\Delta;2^21^2]\\
&\qquad+[\Delta;2^21]+2[\Delta;2^2]+2[\Delta;21^3]+3[\Delta;21^2]+4[\Delta;21]\\
&\qquad+3[\Delta;2]+[\Delta;1^4]+3[\Delta;1^3]+3[\Delta;1^2]+2[\Delta;1]+\Delta)\\
&+[5]\times([\Delta;31]+[\Delta;3]+[\Delta;2^21]+[\Delta;2^2]+[\Delta;21^3]+2[\Delta;21^2]\\
&\qquad+3[\Delta;21]+2[\Delta;2]+[\Delta;1^4]+2[\Delta;1^3]+3[\Delta;1^2]+2[\Delta;1]+\Delta)\\
&+[6]\times([\Delta;21^2]+2[\Delta;21]+[\Delta;2]+[\Delta;1^4]+[\Delta;1^3]+2[\Delta;1^2]\\
&\qquad+2[\Delta;1]+\Delta)\\
&+[7]\times([\Delta;2]+[\Delta;1^3]+[\Delta;1^2]+[\Delta;1]+\Delta)\\
&+[8]\times[\Delta;1]
\end{aligned}
$$

CHAPTER III

COLOUR SUPERALGEBRA AND GENERALIZED QUASISPIN

III.1 INTRODUCTION

In this chapter, we study the embedding of the dynamical algebra U(M/N) of nuclear dynamical supersymmetries in larger algebraic structures. A non-compact $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ graded colour superalgebra SpO(2M/1/2N/O) is identified as a receptacle for various chains containing boson and fermion (super)algebras. The existence of a generalized quasispin algebra is demonstrated and discussed.

The idea of dynamical supersymmetry in nuclei introduced and developed by Balantekin, Bars and Iachello (1981), is a natural outgrowth of the interacting boson model (IBM) of Arima and Iachello (1975, 1979) and the interacting boson-fermion model (IBFM) of Iachello (1980) for collective nuclei states: modes of nuclear excitation in which many protons and neutrons participate simultaneously and coherently. It provides a way to unify the understanding of even-even (even number of protons, even number of neutrons) and even-odd (even number of protons (neutrons), odd number of neutrons (protons)) nuclei which has long been the goal of nuclear structure physics.

The U(6/4) scheme, as the first dynamical supersymmetry to be proposed for nuclei, (Balantekin, Bars and Iachello 1981) has been successfully described in the Pt-Au mass region, which includes both even-even and even-odd nuclei, the low-lying energy levels, electromagnetic transition rates to an accuracy of 10-20% and provided a key to the relation between nuclear transfer reactions in such nuclei. It was in fact claimed to be the first experimental evidence of supersymmetry observed in nature. The success of this model hence stimulated great interest in the field. Various extensions of this scheme and other potentially important dynamical supersymmetry models have been investigated. So far only a small part of the periodic table have been shown to fit certain breaking chains of certain dynamical supersymmetries. The nature of these dynamical supersymmetries are phenomenological, or accidental. The microscopic origin is still unknown.

Recent experimental and theoretical interest in nuclear (and other) dynamical supersymmetries has emphasized the need for the study of the underlying algebraic structures beyond the finite dimensional irreps of compact forms of U(M/N) (Balantekin and Bars 1981, Dondi and Jarvis 1981). A recent extension (Morrison and Jarvis 1985) of the ideas of supersymmetric IBM models to explore the role of fermion pairing (via a seniority scheme) used an intervening OSp(M/N) subalgebra. Although phenomenologically reasonable it suffers (Morrison and Jarvis 1985) from unusual features of non-conservation of nucleon number and non-hermitian interaction V_{BF}, and it has recently been shown (Han, Sun, Zhang and Feng 1985) that the embedding in U(M/N) involves indecomposable representations. Other approaches (Rowe 1985, Moshinsky and Quesne 1970, 1971) to nuclear collective models using non-compact Lie algebras (e.g. Sp(6,R) $\supset$ U(3)) also have supersymmetric enlargements in terms of non-compact OSp(M/N) superalgebras (Han, Liu and Sun 1984); recent attempts to apply supersymmetric IBM models consistently over a range of nuclei (Baake, Reinicke and Gelberg 1986) also suggest larger structures. Finally an eventual microscopic foundation of the IBM ideas will presumably involve infinite dimensional mappings (Talmi 1981).

In this chapter we identify a non-compact $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ colour superalgebra (Rittenberg and Wyler 1978, Lukierski and Rittenberg 1978, Scheunert 1979, Green and Jarvis 1983) SpO(2M/1/2N/O) as a natural receptacle for various chains containing boson and fermion (super) algebras. The first $\mathbb{Z}_2$ corresponds to the usual grading of boson fermion operators, while the additional grading arises from the inclusion of generators both linear (*odd*) and bilinear (*even*) in the boson-fermion realization.

The arrangement of this chapter is as follows. In section III.2, some preliminaries and background is introduced. In section III.3, super creation annihilation operator C_A^{ϵ} is defined from which a $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ graded colour superalgebra SpO(2M/1/2N/0) is constructed. In III.4, we study the various chains of (super) subalgebras contained in the big algebra. It is shown that SpO(2M/1/2N/0) has as (super) subalgebras both the usual fermionic O(2N+1) and bosonic SpO(2M/1) (an alternative to the Heisenberg algebra). Its Fock space realisation (Han, Sun, Zhang and Feng 1985, Han, Liu, Sun 1984) comprises one irrep with just two constituents with respect to SpO(2M/2N). The Casimir invariant of SpO(2m/2N) is a specific linear combination of number operators and suitably defined pairing operators. Indeed they form a generalised quasispin algebra $Sp_{\pm}(2)$ occurring in the $Sp_{\pm}(2) \times SpO_{\pm}(M/N)$ subalgebra of SpO(2M/1/2N/0) (the ± correspond to equivalent choices, interchanged by hermitian conjugation.) Finally, in section III.5, we discuss briefly the significance of the generalized quasispin algebra.

III.2 BASIC CONCEPTS

In this section, we briefly introduce some basic concepts related to the discussions on dynamical supersymmetries in nuclei. Boson and fermion realization of classical Lie algebras and superalgebras are quite well known in physics. It is sketched in section III.2.1. Explicit examples of this realization are given in tables III.2.2 and III.2.3. Nuclei dynamical supersymmetry is reviewed in section III.2.2. Finally in section III.2.3, four compact superidentities for evaluating supercommutators of product operators are given. They play an important role in the construction of superalgebras and particularly in the construction of colour superalgebra C(2;s) which will be discussed in the next section.

III.2.1 Boson Fermion Realization of Classical Lie Algebras and Superalgebras

Lie algebras and Lie superalgebras arise naturally in a many-bodied system through the language of second quantization. The construction of classical Lie algebras using either boson or fermion operators has a long tradition in physics. In recent times classical Lie superalgebras have also been constructed (realized) using both boson and fermion operators. In this section, we briefly review some of the simple constructions.

Consider a system of pure bosons. In the second quantization language, they are described by boson creation annihilation operators $b_i^\dagger$, b_i where i usually stands for a set of quantum numbers characterizing the bosonic states of the system. The following *commutation relations* are satisfied:

$$[b_i^\dagger, b_j] = \delta_{ij} \qquad \text{(III.2.1-1a)}$$

$$[b_i^\dagger, b_j^\dagger] = 0 = [b_i, b_j] \qquad \text{(III.2.2-2b)}$$

A completely symmetric m-boson state in Fock space is obtained by the action of $b_i^\dagger$ on the ground (vacuum) state

$$|m\rangle = b_{i_1}^\dagger b_{i_2}^\dagger \cdots b_{i_m}^\dagger |0\rangle \qquad \text{(III.2.1-2a)}$$

where

$$b_i |0\rangle = 0 \qquad \text{(III.2.1-2b)}$$

Similarly a system of pure fermions are described by fermion creation annihilation operators $f^\dagger_\alpha$, f_α satisfying *anticommutation relations*:

$$\{f^\dagger_\alpha,\ f_\beta\} = \delta_{\alpha\beta} \qquad \text{(III.2.1-3a)}$$

$$\{f^\dagger_\alpha,\ f^\dagger_\beta\} = 0 = \{f_\alpha,\ f_\beta\} \qquad \text{(III.3.1-3b)}$$

The complete antisymmetric n-fermion state is created by

$$|n\rangle = f^\dagger_{\alpha_1}\ f^\dagger_{\alpha_2}\cdots f^\dagger_{\alpha_n}\ |0\rangle \qquad \text{(III.2.1-4a)}$$

where
$$f_\alpha|0\rangle = 0 \qquad \text{(III.2.1-4b)}$$

Further, it is assumed that bosons and fermions are independent, i.e. they commute with one another:

$$[f^\dagger_\alpha,b^\dagger_i] = [f^\dagger_\alpha,b_i] = [f_\alpha,b^\dagger_i] = [f_\alpha,b_i] = 0 \qquad \text{(III.2.1-5)}$$

It is worth noting that although fermion operators themselves satisfy commutation relations, their bilinear products behave like bosons and satisfy commutations. A comparison between properties of boson and fermion operators is given in table III.2.1.

It is well known that the ordinary classical Lie algebras $A(\ell)$, $B(\ell)$, $C(\ell)$, $D(\ell)$ of rank ℓ can be constructed from combinations of linear and bilinear products of either pure boson operators or pure fermion operators. The algebra may be identified by first recognizing the commutative Cartan subalgebra

$$H = \{H_i,\ i = 1,2,\ \cdots,\ \ell\} \qquad \text{(III.2.1-6a)}$$

such that
$$[H_i,\ H_j] = 0 \qquad \text{(III.2.1-6b)}$$

and then comparing the *root vector* $\boldsymbol{\alpha}$ of generator $E_{\boldsymbol{\alpha}}$ defined by the

Table III.2.1 : Comparison between boson and fermion operators

Boson operator b_i $b_i^\dagger$	Fermion operator $f_\alpha, f_\alpha^\dagger$
$b_i b_j = b_j b_i$	$f_\alpha f_\beta = -f_\beta f_\alpha$
$b_i^\dagger b_j^\dagger = b_j^\dagger b_i^\dagger$	$f_\alpha^\dagger f_\beta^\dagger = -f_\beta^\dagger f_\alpha^\dagger$
$b_i b_j^\dagger = b_j^\dagger b_i + \delta_{ij}$	$f_\alpha f_\beta^\dagger = - f_\beta^\dagger f_\alpha + \delta_{\alpha\beta}$
$b_i^\dagger b_j = b_i b_j^\dagger - \delta_{ij}$	$f_\alpha^\dagger f_\beta = - f_\beta f_\alpha^\dagger + \delta_{\alpha\beta}$
$H_i = (1/2)\{b_i^\dagger, b_i\}$	$H_\alpha = (1/2)[f_\alpha^\dagger, f_\alpha]$
$= b_i^\dagger b_i + 1/2$	$= f_\alpha^\dagger f_\alpha - 1/2$
$= b_i b_i^\dagger - 1/2$ (no summation)	$= - f_\alpha f_\alpha^\dagger + 1/2$ (no summation)
$[H_i, b_j] = - \delta_{ij} b_j$	$[H_\alpha, f_\beta] = -\delta_{\alpha\beta} f_\beta$
$[H_i, b_j^\dagger] = \delta_{ij} b_j^\dagger$	$[H_\alpha, f_\beta^\dagger] = \delta_{\alpha\beta} f_\beta^\dagger$
$[H_\alpha, b_j] = 0$	$[H_i, f_\beta] = 0$
$[H_\alpha, b_j^\dagger] = 0$	$[H_i, f_\beta^\dagger] = 0$
$[H_i, b_j b_k] = (-\delta_{ij} - \delta_{ik}) b_j b_k$	$[H_\alpha, f_\beta f_\gamma] = (-\delta_{\alpha\beta} - \delta_{\alpha\gamma}) f_\beta f_\gamma$
$[H_i, b_j b_k^\dagger] = (-\delta_{ij} + \delta_{ik}) b_j b_k^\dagger$	$[H_\alpha, f_\beta f_\gamma^\dagger] = (-\delta_{\alpha\beta} + \delta_{\alpha\gamma}) f_\beta f_\gamma^\dagger$
$[H_i, b_j^\dagger b_k^\dagger] = (\delta_{ij} + \delta_{ik}) b_j^\dagger b_k^\dagger$	$[H_\alpha, f_\beta^\dagger f_\gamma^\dagger] = (\delta_{\alpha\beta} + \delta_{\alpha\gamma}) f_\beta^\dagger f_\gamma^\dagger$

characteristic equation

$$[H, E_{\alpha}] = \alpha E_{\alpha} \tag{III.2.1-7}$$

where $H = (H_1, \cdots, H_{\ell})$, with the known root vector system $\Lambda = \{\alpha | \alpha \in |R^{\ell}\}$ which uniquely characterizes each classical Lie algebra (Judd 1968, Wybourne 1974). Examples of boson and fermion realizations of Lie algebras are given in table III.2.2(a),(b).

Classical Lie superalgebras A(m/n), B(m/n), C(n) and D(m/n) can also be realized in this picture, provided that both fermion and boson operators are used together.

Let $L = L_0 \oplus L_1$ be a classical Lie superalgebra (C.L.S.A.). The *even part* of the algebra, L_0 is a direct sum of two classical Lie algebras C^a and C^b where $C^a, C^b \in \{A(\ell), B(\ell), C(\ell), D(\ell)\}$. Thus we can write

$$L = L_0 \oplus L_1 = (C^a \oplus C^b) \oplus L_1 \tag{III.2.1-8}$$

More explicitly,

$$\begin{aligned} A(m/n) &= A(m/n)_0 \oplus A(m/n)_1 \\ &= (A(m) \oplus A(n)) \oplus A(m/n)_1 \end{aligned} \tag{III.2.1-9a}$$

$$\begin{aligned} B(m/n) &= B(m/n)_0 \oplus B(m/n)_1 \\ &= B(m) \oplus C(n) \oplus B(m/n)_1 \end{aligned} \tag{III.2.1-9b}$$

$$\begin{aligned} C(n) &= C(n)_0 \oplus C(n)_1 \\ &= D(1) \oplus C(n) \oplus C(n)_1 \end{aligned} \tag{III.2.1-9c}$$

$$\begin{aligned} D(m/n) &= D(m/n)_0 \oplus D(m/n)_1 \\ &= D(m) \oplus C(n) \oplus D(m/n)_1 \end{aligned} \tag{III.2.1-9d}$$

Table III.2.2(a) : Boson realization of classical Lie algebras

Group	Algebra	Generators	Root Vector	Number of Generators
$U(\ell)$	$A(\ell)$	$H_i^\dagger = b_i^\dagger b_i + 1/2$	0	ℓ
		$E_{e_i-e_j} = b_i^\dagger b_j$	e_i-e_j	$\underline{\ell(\ell+1)}$
				$\ell(\ell+2)$
$SO(2\ell+1,R)$	$B(\ell)$	$H_i = b_i^\dagger b_i + 1/2$	0	ℓ
		$E_{-e_i} = b_i$	$-e_i$	ℓ
		$E_{e_i} = b_i^\dagger$	e_i	ℓ
		$E_{e_i-e_j} = b_i^\dagger b_j \quad (i\neq j)$	e_i-e_j	$\ell(\ell-1)$
		$E_{e_i+e_j} = b_i^\dagger b_j^\dagger \quad (i\neq j)$	e_i+e_i	$\ell(\ell-1)/2$
		$E_{-e_i-e_j} = b_i b_j \quad (i\neq j)$	$-e_i-e_j$	$\underline{\ell(\ell-1)/2}$
				$\ell(2\ell+1)$
$Sp(2\ell,R)$	$C(\ell)$	$H_i = b_i^\dagger b_i + 1/2$	0	ℓ
		$E_{e_i-e_j} = b_i^\dagger b_j \quad (i\neq j)$	e_i-e_j	$\ell(\ell-1)$
		$E_{e_i+e_j} = b_i^\dagger b_j^\dagger \quad (i\neq j)$	e_i+e_j	$\ell(\ell-1)/2$
		$E_{-e_i-e_j} = b_i b_j \quad (i\neq j)$	$-e_i-e_j$	$\ell(\ell-1)/2$
		$E_{2e_i} = b_i^\dagger b_i^\dagger$	$2e_i$	ℓ
		$E_{-2e_i} = b_i b_i$	$-2e_i$	$\underline{\ell}$
				$\ell(2\ell+1)$
$SO(2\ell,R)$		$H_i = b_i^\dagger b_i + 1/2$	0	ℓ
		$E_{e_i-e_j} = b_i^\dagger b_j \quad (i\neq j)$	e_i-e_j	$\ell(\ell-1)$
		$E_{e_i+e_j} = b_i^\dagger b_j^\dagger \quad (i\neq j)$	e_i+e_j	$\ell(\ell-1)/2$
		$E_{-e_i-e_j} = b_i b_j \quad (i\neq j)$	$-e_i-e_j$	$\underline{\ell(\ell-1)/2}$
				$\ell(2\ell-1)$

Note: b_i, $b_i^\dagger$ for $i = 1,2,\ldots,\ell$ are boson operators, satisfying
$[b_i,b_j^\dagger] = \delta_{ij}$, $[b_i^\dagger,b_j^\dagger] = 0 = [b_i,b_j]$.
No summation over i is assumed here in this table.

Table III.2.2(b) : Fermion realization of classical Lie algebras

Group	Algebra	Generators	Root Vector	Number of Generators
$SU(\ell+1)$	$A(\ell)$	$H_\alpha = f_\alpha^\dagger f_\alpha - 1/2$	0	ℓ
		$E_{e_\alpha - e_\beta} = f_\alpha^\dagger f_\beta$	$e_\alpha - e_\beta$	$\underline{\ell(\ell+1)}$
				$\ell(\ell+2)$
$SO(2\ell+1)$	$B(\ell)$	$H_\alpha = f_\alpha^\dagger f_\alpha - 1/2$	0	ℓ
		$E_{-e_\alpha} = f_\alpha$	$-e_\alpha$	ℓ
		$E_{e_\alpha} = f_\alpha^\dagger$	e_α	ℓ
		$E_{e_\alpha - e_\beta} = f_\alpha^\dagger f_\beta$	$e_\alpha - e_\beta$	$\ell(\ell-1)$
		$E_{e_\alpha + e_\beta} = f_\alpha^\dagger f_\beta^\dagger$	$e_\alpha + e_\beta$	$\ell(\ell-1)/2$
		$E_{-e_\alpha - e_\beta} = f_\alpha f_\beta$	$-e_\alpha - e_\beta$	$\underline{\ell(\ell-1)/2}$
				$\ell(2\ell+1)$
$SO(2\ell)$	$D(\ell)$	$H_\alpha = f_\alpha^\dagger f_\alpha - 1/2$	0	ℓ
		$E_{e_\alpha - e_\beta} = f_\alpha^\dagger f_\beta$	$e_\alpha - e_\beta$	$\ell(\ell-1)$
		$E_{e_\alpha + e_\beta} = f_\alpha^\dagger f_\beta^\dagger$	$e_\alpha + e_\beta$	$\ell(\ell-1)/2$
		$E_{-e_\alpha - e_\beta} = f_\alpha f_\beta$	$-e_\alpha - e_\beta$	$\underline{\ell(\ell-1)/2}$
				$\ell(2\ell-1)$

Note: f_α, $f_\alpha^\dagger$ a = 1,2,...,ℓ are fermion operators satisfying
$\{f_\alpha, f_\beta^\dagger\} = \delta_{\alpha\beta}$, $\{f_\alpha^\dagger, f_\beta^\dagger\} = 0 = \{f_\alpha, f_\beta\}$.

No summation is assumed over α in this table.

For C.L.S.A. Cartan algebra is defined by Kac (1974) to be the Cartan algebra of L_0, which is the direct sum of Cartan algebras of classical Lie algebra C^a and C^b. If H^a, H^b are Cartan subalgebras for C^a and C^b respectively, i.e.

$$H^a = (H_1{}^a, \cdots, H_m{}^a) \subset C^a, \quad [H^a, E_\alpha] = \alpha E_\alpha \qquad \text{(III.2.1-10a)}$$

$$H^b = (H_1{}^b, \cdots, H_n{}^b) \subset C^b, \quad [H^b, E_\beta] = \beta E_\beta \qquad \text{(III.2.1-10b)}$$

where root space $\Delta^a = \{\alpha | \alpha \in \mathbb{R}^m\}$ and $\Delta^b = \{\beta | \beta \in \mathbb{R}^n\}$ are spanned by the orthonormal basis

$$\{\epsilon_1, \epsilon_2, \cdots, \epsilon_m\} \quad \text{for } C^a \qquad \text{(III.2.1-11a)}$$

$$\{\delta_1, \delta_2, \cdots, \delta_n\} \quad \text{for } C^b \qquad \text{(III.2.1-11b)}$$

respectively, then the Cartan subalgebra for $L_0 = C^a \oplus C^b$ is

$$\begin{aligned} H &= (H_1{}^a, \cdots, H_m{}^a, H_1{}^b, \cdots, H_n{}^b) \\ &= (H_1, \cdots, H_m, H_{m+1}, \cdots, H_{m+n}) \end{aligned} \qquad \text{(III.2.1-11c)}$$

satisfying

$$H \subset L_0 \subset L \qquad \text{(III.2.1-12)}$$

$$[H, L_0] \subset L_0 \qquad \text{(III.2.1-13)}$$

$$[H, L_1] \subset L_1 \qquad \text{(III.2.1-14)}$$

and

$$[H, E_\sigma] = \sigma E_\sigma \qquad \sigma \in R^{m+n} \qquad \text{(III.2.1-15)}$$

The root space of C.L.S.A. is

$$\Delta = \Delta_0 \oplus \Delta_1 \qquad \text{(III.2.1-16a)}$$

spanned by

$$\{\epsilon_1, \cdots, \epsilon_m, \delta_1, \cdots, \delta_n\} \quad \in \mathbb{R}^{m+n} \qquad \text{(III.2.1-16b)}$$

where Δ_0 contains the *even roots* α (either in terms of ϵ_α or δ_i) corresponding to the even generators $E_\alpha \in L_0$ while Δ_1 the *odd roots* β corresponding to odd generators $E_\beta \in L_1$.

Standard root systems for C.L.S.A. given by Kac (1974) provide us with defining relations of the superalgebras. Corresponding to each root vector, we have a generator expressed as linear or bilinear products of boson and/or fermion operators. The actual construction proceeds as follows. First of all, we construct generators of C^a and C^b, one with fermions and the other with bosons (or vice versa). The direct sum of C^a and C^b corresponds to L_0. Next, we construct the odd part of the superalgebra using mixed operators of the form $f^\dagger b^\dagger$, $f^\dagger b$, $fb^\dagger$, fb, or f.

Explicit boson-fermion realizations for C.L.S A. are given in table III.3.2. Again, they are not the only realizations possible.

Table III.2.3 : **Boson-fermion realization of classical Lie superalgebras**

C.L.S.A.			Generators	Root vectors
A(m/n)	$A(m/n)_0$	U(m)	H_α, $f^\dagger_\alpha f_\beta$	0, $\epsilon_\alpha-\epsilon_\beta$
		U(n)	H_i, $b^\dagger_i b_j$	0, $\delta_i-\delta_j$
	$A(m/n)_1$		$f^\dagger_\alpha b_i$ $f_\alpha b^\dagger_i$	$\epsilon_\alpha-\delta_i$, $-\epsilon_\alpha+\delta_i$
B(m/n)	$B(m/n)_0$	SO(2n+1)	H_α, $f^\dagger_\alpha f^\dagger_\beta$, $f^\dagger_\alpha f_\beta$, $f_\alpha f_\beta$, $f^\dagger_\alpha$, f_α	0, $\pm\epsilon_\alpha\pm\epsilon_\beta$, $\pm\epsilon_\alpha$
		Sp(2n,R)	H_i, $b^\dagger_i b^\dagger_j$, $b^\dagger_i b_j$, $b_i b_j$, $b^\dagger_i b^\dagger_i$, $b_i b_i$	0, $\pm\delta_i\pm\delta_j$, $\pm 2\delta_i$
	$B(m/n)_1$		$f^\dagger_\alpha b^\dagger_i$, $f^\dagger_\alpha b_i$, $f_\alpha b^\dagger_i$, $f_\alpha b_i$, $b^\dagger_i$, b_i	$\pm\epsilon_\alpha\pm\delta_i$, $\pm\delta_i$
C(n)	$C(n)_0$	SO_2	$H = f^\dagger_1 f_1 - 1/2$	0
		Sp(2n,R)	H_i, $b^\dagger_i b^\dagger_j$, $b^\dagger_i b_j$, $b_i b_j$, $b^\dagger_i b^\dagger_i$, $b_i b_i$	$\pm\delta_i\pm\delta_j$, $\pm 2\delta_i$
	$C(n)_1$		$f^\dagger_1 b^\dagger_i$, $f^\dagger_1 b_i$, $f_1 b^\dagger_i$, $f_1 b_i$	$\pm\epsilon_1\pm\delta_i$

Table III.2.3 continued on next page

Table III.2.3 continued

C.L.S.A.			Generators	Root vectors
D(m/n)	$D(m/n)_0$	SO(2m)	H_α, $f^\dagger_\alpha f^\dagger_\beta$, $f^\dagger_\alpha f_\beta$, $f_\alpha f_\beta$	0, $\pm\epsilon_\alpha\pm\epsilon_\beta$
		Sp(2n,R)	H_i, $b^\dagger_i b^\dagger_j$, $b^\dagger_i b_j$, $b_i b_j$, $b^\dagger_i b^\dagger_i$, $b_i b_i$	0, $\pm\delta_i\pm\delta_j$, $\pm 2\delta_i$
	$D(m/n)_1$		$f^\dagger_\alpha b^\dagger_i$, $f^\dagger_\alpha b_i$, $f_\alpha b^\dagger_i$, $f_\alpha b_i$	$\pm\epsilon_\alpha\pm\delta_i$

Note: Fermion index $\alpha,\beta=1,2,\cdots,m$; boson index $i,j=1,2,\cdots,n$.
Root vectors $\epsilon_\alpha \in R^m$, $\epsilon_\alpha+\epsilon_\beta \in R^m$, $\delta_i \in R^n$, $\delta_i+\delta_j \in R^n$ $\epsilon_\alpha+\delta_i \in R^{m+n}$, and in $\pm\epsilon_\alpha\pm\delta_i$ say, all possible combinations of signs are teken. H_i and H_α are as defined in Table III.2.1.

III.2.2 Review of Dynamical Symmetries in Nuclei

Dynamical symmetry has occurred in different fields of physics, for example in elementary particle and atomic theory. It is a very useful tool in dealing with many-body problems where the interactions are too complicated, or when the origin and nature of them are unknown. The main idea is to propose a chain of groups (or algebras):

$$G_0 \supset G_1 \supset G_2 \supset \cdots G_n \qquad \text{(III.2.2-1)}$$

and express the Hamiltonian which contains all information about interactions, in terms of Casimir operators $C(G_i)$ of the group chain (one body interaction corresponds to first order Casimir operator, two body interaction to second order Casimir operator, etc.) as

$$H = k_0C(G_0) + k_1C(G_1) + \cdots + k_nC(G_n) \qquad \text{(III.2.2.-2)}$$

The value of Casimir invariants, hence the energy levels, depends on the state the system occupies which can be classified by the set of irreps of the group chain, denoted by irrep labels or quantum numbers. k_0, $k_1, \cdots, k_n$ are empirical parameters. Suppose we can determine the values of these parameters so that the experimentally obtained energy levels can be expressed reasonably well by the analytic formula (III.2.2.-2), then we may say that the system processes the dynamical symmetry of the group chain (III.2.2-1).

The largest group in the chain, G_0, which is sometimes called the limiting symmetry (or parent, or source group) represents the highest symmetry of the system. If the interactions were not there all states within the same G_0 multiplet (with the same G_0 irrep label) would have the same energy. G_1 is a subgroup of G_0. It represents interaction of some sort which reduces (breaks) the symmetry from G_0 to a lower symmetry G_1. In other words the effect of the presence of G_1 is to split a multiplet of G_0 into several multiplets of G_1, hence a single energy level to several sublevels. Similarly, G_2 breaks G_1 symmetry and so on, until we reach G_n which is the lowest symmetry of the system. For nuclei systems G_n is the three dimensional rotation group SO(3). In this description, interactions are interpreted as a process of successive symmetry breaking.

The traditionally used groups in dynamical symmetry studies are ordinary groups, hence they are termed *ordinary dynamical symmetry*. If instead, G_0 is a supergroup (its subgroups G_i may be ordinary or supergroups), then naturally enough it is called *dynamical supersymmetry*.

Dynamical symmetry in nuclei was first introduced by Arima and Iachello in IBM and IBFM. The most crucial step is to replace the geometric shape variables which were normally used to describe collective phenomenon of nuclei, for instance in the liquid drop model of Bohr and Mottelson (1975), with the boson fermion variables described mathematically by the creation and annihilation operators. The advantage is obvious as seen in section III.2-1. Such a system of bosons and fermions have very rich algebraic structures, hence provides a natural framework for the study of symmetries.

A nucleus contains both protons and neutrons (nucleons) which are fermionic particles. The strong nuclear force tends to pair nucleons together. In IBM, proposed for even-even nuclei, the paired valence nucleons counted from the nearest closed shell (which is regarded as ground or vacuum state $|0\rangle$ of the system), are replaced by effective bosons with spin 0 or 2 (s and d bosons) with no distinction between neutrons and protons or particles above and holes below has been made. The highest symmetry group is U(6) generated by $\{b^{\dagger}_{\ell_1 m_1} b_{\ell_2 m_2}\}$ where $\ell_1, \ell_2 = 0$ or 2, and only three chains of symmetry breaking exist, each corresponding to a well defined physical limit and suitable for describing certain even-even nuclei

$$U(6)\begin{cases}\supset SU_3 \supset O(3) & \text{(rotational limit)} \qquad \text{(III.2.2-3a)}\\ \supset O(6) \supset O(5) \supset O(3) & \text{(γ-unstable limit)} \qquad \text{(III.2.2-3b)}\\ \supset U(5) \supset O(5) \supset O(3) & \text{(vibrational limit).} \qquad \text{(III.2.2-3c)}\end{cases}$$

In IBFM, proposed for even-odd or odd-even nuclei, the paired nucleons are again replaced by bosons while the unpaired nucleons by effective fermions, whose spin is related to the angular momentum of the ground state (closed shell) of the nucleus. The maximum symmetry for this model is U(6) × U(4). The possible chains of symmetry breaking is far more than the IBM case. In the IBFM model boson states occur in multiplet of U(6) while fermion states in multiplet of U(4) and they do not mix due to the absence of operators in the algebra which turns bosons into fermions and vice versa. This model has been successful in describing certain even-odd nuclei.

Nuclei dynamical supersymmetry (or supersymmetric IBM) U(6/4) is a natural extension of IBM and IBFM. The odd generators of U(6/4) are of the form $b^\dagger f^\dagger$, $b^\dagger f$, bf, $f^\dagger b$ which links bosons with fermions, hence even-even nuclei with even-odd (odd-even) nuclei. Bosons and fermions occur in the same multiplet for this model.

At present, many different supersymmetries are being proposed for nuclei. This involves the construction of possible group chains of a supergroup G_0 (G_0 is usually taken to be U(M/N) following the success of U(6/4)), calculation of the energy levels using analytic expression (III.2.2-2) and other physical quantities such as electromagnetic transition rates. Experimental tests of the results are also being carried out.

III.2.3 Operator Identities and Superidentities

One of the frequent tasks encountered in the second quantization realization (or harmonic oscillator realization) of Lie (super)algebras is to evaluate supercommutators (commutators or anticommutators) involving products of boson and/or fermion operators. The following 'superidentities' are very useful:

$$\langle A,BC\rangle = \langle A,B\rangle C + (-)^{(A)\cdot(B)}B\langle A,C\rangle \qquad \text{(III.2.3-1)}$$

$$\langle AB,C\rangle = A\langle B,C\rangle + (-)^{(B)\cdot(C)}\langle A,C\rangle B \qquad \text{(III.2.3-2)}$$

$$\begin{aligned}\langle AB,CD\rangle = {} & (-)^{(B)\cdot(C)}\langle A,C\rangle BD + (-)^{(A+B)\cdot(C)+(B)\cdot(D)}C\langle A,D\rangle B \\ & + A\langle B,C\rangle D + (-)^{(A+B)\cdot(C)}CA\langle B,D\rangle \qquad \text{(III.2.3-3)}\end{aligned}$$

$$\begin{aligned}\langle AB,CD\rangle = {} & (-)^{(B)\cdot(C+D)}\langle A,C\rangle DB + (-)^{(A+B)\cdot(C)+(B)\cdot(D)}C\langle A,D\rangle B \\ & + A\langle B,C\rangle D + (-)^{(B)\cdot(C)}AC\langle B,D\rangle \qquad \text{(III.2.3-4)}\end{aligned}$$

where $\langle A,B\rangle = AB - (-)^{(A)\cdot(B)}BA$, $(A+B) = (A) + (B), (A), (B) \in \mathbb{Z}_2$ is the grading vector of operator A,B, $(A)\cdot(B)$ is the ordinary scalar product of vector (A) and (B).

If we assign all possible $\mathbb{Z}_2$ grading for operator A, B, C and D then each superidentity gives rise to several ordinary operator identities. For example, corresponding to the superidentity (III.2.2-1), we have four ordinary identities, namely

$$[A,BC] = [A,B]C + B[A,C], \qquad \text{(III.2.3-5a)}$$

$$[A,BC] = \{A,B\}C-B\{A,C\}, \qquad \text{(III.2.3-5b)}$$

$$\{A,BC\} = [A,B]C + B\{A,C\}, \qquad \text{(III.2.3-5c)}$$

$$\{A,BC\} = \{A,B\}C - B[A,C]. \qquad \text{(III.2.3-5d)}$$

Similarly, (III.2.3-2) corresponds to four distinct ordinary operator identities while (III.2.2-3 and 4) each corresponds to ten ordinary identities, e.g.

$$[AB,CD] = [A,C]BD + C[A,D]B + A[B,C]D + CA[B,D] \qquad \text{(III.2.3-6a)}$$

$$[AB,CD] = -\{A,C\}BD - C\{A,D\}B + A\{B,C\}D + CA\{B,D\} \qquad \text{(III.2.3-6b)}$$

$$\{AB,CD\} = \{A,C\}BD - C[A,D]B + A[B,C]D - CA[B,D] \qquad \text{(III.2.3-6c)}$$

$$\{AB,CD\} = [A,C]BD - C[A,D]B + A[B,C]D + CA\{B,D\}, \text{ etc.} \qquad \text{(III.2.3-6d)}$$

Proofs of the superidentities are straight forward but tedious, hence will not be presented here.

The $\mathbb{Z}_2$ grading for boson and fermion operators are defined as follows

$$(b^\dagger) = (b) = 0 \qquad \text{(III.2.3-7a)}$$

$$(f^\dagger) = (f) = 1 \qquad \text{(III.2.3-7b)}$$

i.e. $b^\dagger$, b are *even* operators while $f^\dagger$ and f are all *odd* operators. The grading of a product operator is defined to be the sum of grading of each individual operator. Thus

$$(fb) = (f) + (b) = 1 + 0 = 1 \text{ (odd)}, \qquad \text{(III.2.3-7c)}$$

$$(ff) = (f) + (f) = 1 + 1 = 0 \text{ (even)}, \qquad \text{(III.2.3-7d)}$$

$$(bb) = (b) + (b) = 0 + 0 = 0 \text{ (even)}. \qquad \text{(III.2.3-7e)}$$

Example: Using the superidentity (III.2.3-1) and the grading defined in (III.2.3-7) we easily obtain

$$\{f_\alpha, f_\beta{}^\dagger b_i\} = \{f_\alpha, f_\beta{}^\dagger\}b_i - f_\beta{}^\dagger[f_\alpha, b_i] = \delta_{\alpha\beta} b_i \quad \text{by} \qquad \text{(III.2.1-3)}$$

Sometimes, it is necessary to take several copies of boson or fermion operators to construct an algebra (Bars, 1983). Let them be denoted as $b_i^\dagger(r)$, $b_i(r)$, $f_\alpha{}^\dagger(r)$, $f_\alpha(r)$ where the extra label r is used to distinguish different copies. Operators with different r values will necessarily commute. In this case, the grading can be redefined as $\mathbb{Z}_2\oplus\mathbb{Z}_2\cdots\oplus\mathbb{Z}_2$ vectors (r copies) and the superidentities will still be valid. For example, let r = 2. We have

$$(b(1)^{\dagger}) = (b(1)) = (0,0) \qquad \text{(III.2.3-9a)}$$

$$(b(2)^{\dagger}) = (b(2)) = (0,0) \qquad \text{(III.2.3-9b)}$$

$$(f(1)^{\dagger}) = (f(1)) = (1,0) \qquad \text{(III.2.3-9c)}$$

$$(f(2)^{\dagger}) = (f(2)) = (0,1) \qquad \text{(III.2.3-9d)}$$

where (0,0), (1.0), (0,1) $\in \mathbb{Z}_2\oplus\mathbb{Z}_2$. Using the above grading and identity (III.2.3-3), it is very easy to show for instance,

$$\begin{aligned}
&[b(1)_i^{\dagger}\, b(1)_i,\; f(2)_{\alpha}^{\dagger}\, b(1)_j^{\dagger}] \\
&= [b(1)_i^{\dagger}, f(2)_{\alpha}^{\dagger}]\, b(1)_i\, b(1)_j^{\dagger} + f(2)_{\alpha}^{\dagger}\, [b(1)_i^{\dagger}, b(1)_j^{\dagger}]\, b(1)_i \\
&+ b(1)_i^{\dagger}\, [b(1)_i, f(2)_{\alpha}^{\dagger}] b(1)_j^{\dagger} + f(2)_{\alpha}^{\dagger}\, b(1)_i^{\dagger}\, [b(1)_i, b(1)_j^{\dagger}] \\
&= \delta_{ij}\, f(2)_{\alpha}^{\dagger}\, b(1)_i^{\dagger}
\end{aligned}$$

The superidentities are useful in section III.3 where we construct colour superalgebra and its subalgebras.

III.3 $\mathbb{Z}_2\oplus\mathbb{Z}_2$ GRADED COLOUR SUPERALGEBRA SPO(2M/1/2N/0)

In section III.2.2 we have shown simple examples of boson-fermion realizations of classical Lie algebras and superalgebras and the importance of these algebras in the study of dynamical symmetries of nuclei. In this section we want to explore further the algebraic structures of the mixed boson-fermion system. A non-compact $\mathbb{Z}_2\oplus\mathbb{Z}_2$ graded colour superalgebra constructed out of boson and fermion operators, both linear and bilinear, is identified as a receptacle for various chains of (super)subalgebras including the dynamical algebra U(M/N) of nuclear supersymmetries currently under study by various physicists. This larger algebraic structure also contains a generalized quasispin algebra which is particularly interesting.

Definition of $\mathbb{Z}_2\oplus\mathbb{Z}_2\oplus\cdots\oplus\mathbb{Z}_2$ (n copies) graded colour superalgebra C(n;s) is given in section III.3.1. After the gradation of operators is carefully chosen we show that the system of linear and bilinear boson fermion operators satisfy closure under the supercommutation relation of C(2;s) and form a $\mathbb{Z}_2\oplus\mathbb{Z}_2$ graded colour superalgebra which we shall denote as

$$\mathrm{SpO}(2M_{(0,0)}/1_{(0,1)}/2N_{(1,0)}/0_{(1,1)})$$

where 2M,1,2N,0 refer to the dimension and the subscripts to $\mathbb{Z}_2\oplus\mathbb{Z}_2$ double grading, of the underlying vector spaces $V_{(0,0)}$, $V_{(0,1)}$, $V_{(1,0)}$ and $V_{(1,1)}$ respectively, of the superalgebra. The symbol SpO reminds us that the maximal Lie superalgebra contained is the non-compact SpO(2M/2N) where the bosons span Sp(2M) and the fermions span SO(2N).

III.3.1 Colour Superalgebra C(n;s)

Following J. Lukierski and V. Rittenberg (1978), we define the colour superalgebra as follows. Let $L = \sum_\alpha \oplus L_\alpha$ be a *graded vector space*, $\alpha = (\alpha_1,\cdots,\alpha_n)$ be an n-component *grading vector* whose components are integer numbers modula two. i.e. $\alpha \in \mathbb{Z}_2\oplus\cdots\oplus\mathbb{Z}_2$ (n copies) $\alpha_i \in \mathbb{Z}_2 = \{0,1\}$. The 2^n vectors α form a 'colour space' with n colours. Let the *supercommutator* $\langle,\rangle$ be a bilinear map of $L \times L \to L$ which satisfies the following three conditions:

(i) closure: $\langle L_\alpha, L_\beta\rangle \in L_{\alpha+\beta}$ (III.3.1-1)

(ii) symmetry: $\langle L_\alpha, L_\beta\rangle = -(-)^{(\alpha)\cdot(\beta)} \langle L_\beta, L_\alpha\rangle$ (III.3.1-2)

(iii) generalized Jacobi identity

$$\langle L_{i\alpha}, \langle L_{j\beta}, L_{k\gamma}\rangle\rangle (-)^{(\alpha)\cdot(\gamma)}$$
$$+ \text{ cyclic permutation} = 0 \qquad \text{(III.3.1-3)}$$

where

$$(\alpha)\cdot(\beta) = \alpha_1\beta_1 + \alpha_2\beta_2 \cdots + \alpha_n\beta_n \qquad \text{(III.3.1-4)}$$

is the Euclidean symmetric scalar product of the grading vectors. Then $\{L, \langle,\rangle\}$ forms a $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \cdots \oplus \mathbb{Z}_2$ (n copies) *graded colour superalgebra* denoted by C(n;s). (Note: s used here is to emphasize the symmetric nature of the scalar product between grading vectors. There exists another colour superalgebra C(n;a) where the scalar product is defined to be symplectic antisymmetric). In the special case where n = 1, colour superalgebra reduces to ordinary Lie superalgebra ($\mathbb{Z}_2$ graded)

$$L = L_0 \oplus L_1 \qquad \text{(III.3.1-5a)}$$

with

$$[L_0, L_0] \subset L_0 \qquad \text{(III.3.1-5b)}$$

$$[L_0, L_1] \subset L_1 \qquad \text{(III.3.1-5c)}$$

$$\{L_1, L_1\} \subset L_0. \qquad \text{(III.3.1-5d)}$$

The particular case we are interested in is the $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ colour superalgebra C(2;s). It contains four sectors

$$L = L_{(0,0)} \oplus L_{(0,1)} \oplus L_{(1,0)} \oplus L_{(1,1)} \qquad \text{(III.3.1-6)}$$

and the supercommutation $\langle,\rangle$ is explicitly spelt out as in table III.3.1.

Table III.3.1 : Supercommutation relation of $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ graded colour superalgebra C(2;s)

$\langle L_\alpha, L_\beta \rangle$	$L_{(0,0)}$	$L_{(0,1)}$	$L_{(1,0)}$	$L_{(1,1)}$
$L_{(0,0)}$	$[L_{(0,0)}, L_{(0,0)}] \subset L_{(0,0)}$	$[L_{(0,0)}, L_{(0,1)}] \subset L_{(0,1)}$	$[L_{(0,0)}, L_{(1,0)}] \subset L_{(1,0)}$	$[L_{(0,0)}, L_{(1,1)}] \subset L_{(1,1)}$
$L_{(0,1)}$		$\{L_{(0,1)}, L_{(0,1)}\} \subset L_{(0,0)}$	$[L_{(0,1)}, L_{(1,0)}] \subset L_{(1,1)}$	$\{L_{(0,1)}, L_{(1,1)}\} \subset L_{(1,0)}$
$L_{(1,0)}$			$\{L_{(1,0)}, L_{(1,0)}\} \subset L_{(0,0)}$	$\{L_{(1,0)}, L_{(1,1)}\} \subset L_{(0,1)}$
$L_{(1,1)}$				$[L_{(1,1)}, L_{(1,1)}] \subset L_{(0,0)}$

Note: $L = \sum_\alpha L_\alpha$, $\langle L_a, L_\beta \rangle = L_\alpha L_\beta - (-)^{\alpha_1\beta_1+\alpha_2\beta_2} L_\beta L_\alpha \subset L_{\alpha+\beta}$, $\alpha, \beta \in \mathbb{Z}_2 \oplus \mathbb{Z}_2$.

III.3.2 Boson-Fermion Realization of $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ Graded Colour Superalgebra C(2;s)

In order to study a complex algebraic structure, some care must be exercised in establishing a concise and consistent notation. To reflect the fact that bosons and fermions are treated on an equal footing in supersymmetries, let us denote boson and fermion operators collectively by the super operator C_A^ϵ where ϵ = 0, or 1 refers to annihilation or creation operators and A = a or α refers to boson or fermion operators respectively.

Let B be the Latin *boson index set*:

$$B = \{a | a = \ell_a m_a,\ m_a = \pm\ell_a,\ \pm(\ell_a - 1), \cdots 0,\ \ell_a \in \mathbb{Z}\} \tag{III.3.2-1}$$

F be the Greek *fermion index set*:

$$F = \{\alpha | \alpha = j_\alpha m_\alpha,\ m_\alpha = \pm j_\alpha,\ \pm(j_\alpha - 1), \cdots \pm 1/2,\ j_\alpha \in \mathbb{Z} - \tfrac{1}{2}\} \tag{III.3.2-2}$$

and I, the *total index set*:

$$I = \{A | A = a \text{ or } \alpha\} = B \cup F \tag{III.3.2-3}$$

or

$$I = \{A | A = j_A m_A,\ m_A = \pm j_A,\ \pm(j_A - 1), \cdots, 1 \text{ or } \tfrac{1}{2},\ j_A \in \mathbb{Z} \text{ or } \mathbb{Z} - \tfrac{1}{2}\} \tag{III.3.2-4}$$

where $\mathbb{Z}$ and $\mathbb{Z}-\frac{1}{2}$ is the set of positive integers and half integers respectively. For convenience, let us define $-A$ as

$$-A = j_A - m_A \tag{III.3.2-5}$$

Consider the transformation properties of C_A^ϵ for $\epsilon = 1$ and 0. The supercreation operator C_A^1 satisfies

$$[J_\pm, C_A^1] = \{(J_A \mp m_A)(j_A \pm m_A + 1)\}^{1/2} C^1_{j_A\ m_A \pm 1} \qquad \text{(III.3.2-6)}$$

$$[J_z, C_A^1] = m_A\ C_A^1 \qquad \text{(III.3.2-7)}$$

hence the $(2J_A + 1)$ operators C_A^1 for fixed j_A and $m_A = -j_A, -j_A+1, \cdots, j_A$ transform as an irreducible spherical tensor of rank j_A under rotation of ordinary space. The modified superannihilation operators defined by

$$C_A^o = (-)^{j_A \mp m_A} (C^1_{-A})^\dagger \qquad \text{(III.3.2-8a)}$$

$$C_a^o = \tilde{b}_a = (-)^{j_a \mp m_a}\ b_{-a} \qquad \text{(III.3.2-8b)}$$

$$C_\alpha^o = \tilde{f}_\alpha = (-)^{j_\alpha \mp m_\alpha}\ f_{-\alpha} \qquad \text{(III.3.2-8c)}$$

transform in the same way as C_A^1, hence they also form components of a rank j_A tensor of SO_3. Note that either choice of −, + sign in the phase factors is acceptable and we will keep track of both in the subsequent discussions.

The number of distinct boson and fermion indices is given by

$$M = \sum_{\ell}(2\ell+1) = \sum_{j_a}(2j_a + 1) \qquad \text{(III.3.2-9a)}$$

$$N = \sum_{j}(2j+1) = \sum_{j_a}(2j_\alpha + 1) \qquad \text{(III.3.2-9b)}$$

and the total number of superoperators C_A^ϵ is

$$2M + 2N \qquad \text{(III.3.2-10)}$$

The actual selection of ℓ and j values is determined by concrete physical problems concerned. There is no need at this stage to impose any restrictions on it since we are interested in structures as general as possible.

$\{C^{\sigma}_{A},\ A \in I,\ \sigma \in \mathbb{Z}_2\}$ forms a well known $\mathbb{Z}_2$ graded vector space with the usual σ independent grading defined by (III.2.3-7):

$$(C^{\sigma}_{a}) = (a) = 0 \qquad \text{(III.3.2-11a)}$$

$$(C^{\sigma}_{\alpha}) = (\alpha) = 1 \qquad \text{(III.3.2-11b)}$$

i.e. bosons and fermions belong to *even* and *odd* subspaces of the same $\mathbb{Z}_2$ graded vector space respectively,

$$V(2M_{(0)}/2N_{(1)}) = V(2M)_{(0)} \oplus V(2N)_{(1)} \qquad \text{(III.3.2-12a)}$$

where

$$V(2M)_{(0)} = \{C^{\sigma}_{a}\} = \{b^{\dagger}_{a},\ \tilde{b}_{a}\} \qquad \text{(III.3.2-12b)}$$

$$V(2N)_{(1)} = \{C^{\sigma}_{\alpha}\} = \{f^{\dagger}_{\alpha},\ \tilde{f}_{\alpha}\} \qquad \text{(III.3.2-12c)}$$

The commutation and anticommutations among the boson and fermion operators may be compactly written in terms of supercommutator $\langle,\rangle$ as

$$\langle C^{\sigma}_{A},\ C^{\tau}_{B}\rangle = G^{\sigma\tau}_{AB}\ , \qquad \text{(III.3.2-13)}$$

$$\langle C^{\sigma}_{A},\ C^{\tau}_{B}\rangle = C^{\sigma}_{A}\, C^{\tau}_{B} - (-)^{(A)(B)} C^{\tau}_{B}\, C^{\sigma}_{A}, \qquad \text{(III.3.2-14)}$$

where $A,\ B \in I,\ \sigma,\tau \in \mathbb{Z}_2,\ (A),\ (B) \in \mathbb{Z}_2$ is the grading vector for C^{ϵ}_{A} and C^{τ}_{β} respectively, $(-)^{(A)(B)}$ is the commutation factor which determines whether $\langle,\rangle$ is an commutator $[,]$ or an anticommutator $\{,\}$. $(G^{\sigma\tau}_{AB})$ is the $\mathbb{Z}_2$ graded antisymmetric *metric tensor* on the $\mathbb{Z}_2$ graded vector space $V(2M_{(0)}/2N_{(1)})$. Explicit form of $G^{\sigma\tau}_{AB}$ and is given in table III.3.2. We note that

Table III.3.2 : $\mathbb{Z}_2$ graded metric tensor $G^{\sigma\tau}_{AB}$

$G^{\sigma\tau}_{AB} = \langle C^{\sigma}_{A}, C^{\tau}_{B}\rangle$	C^{0}_{b}	C^{1}_{b}	C^{0}_{β}	C^{1}_{β}
C^{0}_{a}	0	g_{ab}	0	0
C^{1}_{a}	$-g_{ab}$	0	0	0
C^{0}_{α}	0	0	0	$g_{\alpha\beta}$
C^{1}_{α}	0	0	$-g_{\alpha\beta}$	0

where

$$g_{ab} = G^{01}_{ab} = [C^{0}_{a}, C^{1}_{b}] = (-)^{\ell_a \mp m_a} \delta_{-ab},$$

$$g_{\alpha\beta} = G^{01}_{\alpha\beta} = \{C^{0}_{\alpha}, C^{1}_{\beta}\} = (-)^{j_\alpha \mp m_\alpha} \delta_{-\alpha\beta},$$

$\sigma,\tau \in \mathbb{Z}_2$; $A,B \in I'$.

$$G^{\epsilon\tau}_{AB} = 0 \text{ if } \epsilon = \tau \text{ or } (A+B) = (A) + (B) = 1 \qquad \text{(III.3.2-15)}$$

in accordance with equation (III.2.1-1b), (III.2.1-3b) and (III.2.1-5) respectively. The metric tensor $G^{\sigma\tau}_{AB}$ permits considerable compactification of supercommutation calculations.

Properties of the $\mathbb{Z}_2$ graded metric tensor $G^{\sigma\tau}_{AB}$ is summarized. For convenience, let us define g_{AB}, g_{ab}, $g_{\alpha\beta}$ and the matrix $\theta^{\sigma\tau}$ as follows

$$g_{AB} = G^{01}_{AB} = \langle C^0_A, C^1_B \rangle, \qquad \text{(III.3.2-16a)}$$

$$g_{AB} = g_{ab} \oplus g_{\alpha\beta} \qquad \text{III.3.2-16b)}$$

$$\theta^{\sigma\tau} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \qquad \sigma,\tau = 0,1 \qquad \text{(III.3.2-17)}$$

Then

$$G^{\sigma\tau}_{AB} = \theta^{\sigma\tau} g_{AB} \qquad \text{with } A,B \in I,\ \sigma,t \in \mathbb{Z}_2 \qquad \text{(III.3.2-18)}$$

Explicitly,

$$g_{AB} = (-)^{J_A \mp m_A}\ \delta_{-AB} \qquad \text{(III.3.2-19a)}$$

$$g_{ab} = (-)^{j_a \mp m_a}\ \delta_{-ab} \qquad \text{(III.3.2-19b)}$$

$$g_{\alpha\beta} = (-)^{j_\alpha \mp m_\alpha}\ \delta_{-\alpha\beta} \qquad \text{(III.3.2-19c)}$$

and the inverses are

$$g^{AB} = (-1)^{(A)(B)} (-1)^{j_A \mp m_A}\ \delta_{-AB} \qquad \text{(III.3.2-20a)}$$

$$g^{ab} = (-1)^{j_a \mp m_a}\ \delta_{-ab} \qquad \text{(III.3.2-20b)}$$

$$g^{\alpha\beta} = -(-1)^{j_\alpha \mp m_\alpha}\ \delta_{-\alpha\beta} \qquad \text{(III.3.2-20c)}$$

$$\theta_{\sigma\tau} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \qquad \sigma,\tau = 0,1 \qquad \text{(III.3.2-20d)}$$

such that $g^{AB}\ g_{BC} = \delta^A_C$, $g^{ab}\ g_{bc} = \delta^a_C$, $g^{\alpha\beta}\ g_{\beta\gamma} = \delta^\alpha_\gamma$.

Symmetry properties:

$$G^{\sigma\tau}_{AB} = (-)^{(A)+(B)+1}\ G^{\tau\sigma}_{BA} \qquad \text{(graded antisymmetric)} \qquad \text{(III.3.2-21a)}$$

$$g_{AB} = (-)^{(A)\cdot(B)}\, g_{BA} \qquad \text{(graded symmetric)} \qquad \text{(III.3.2-21b)}$$

$$g_{ab} = g_{ba} \qquad \text{(symmetric)} \qquad \text{(III.3.2-21c)}$$

$$g_{\alpha\beta} = -g_{\beta\alpha} \qquad \text{(antisymmetric)} \qquad \text{(III.3.2-21d)}$$

and

$$g^{BA} = (-)^{(A)\cdot(B)}\, g_{AB} \qquad \text{(III.3.2-22a)}$$

$$g^{ba} = g_{ab} \qquad \text{(III.3.2-22b)}$$

$$g^{\beta\alpha} = -g_{\beta\alpha} \qquad \text{(III.3.2-22c)}$$

The special metric tensors g_{AB}, g^{AB}, etc., are important in the construction of subalgebras of graded colour superalgebras.

The $\mathbb{Z}_2$ graded vector space $V(2M_{(0)}/2N_{(1)})$ of supercreation annihilation operators may be extended to a doubly graded $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ vector space by introducing the identity operator $C^{\epsilon}_{*} = e$ as follows. The index set $I = B \cup F$ is extended to I' where

$$I' = B \cup F \cup \{*\} \qquad \text{(III.3.2-23)}$$

The extended grading is defined as

$$(A) \rightarrow ((A),0) \qquad \text{(III.3.2-24)}$$

thus

$$(C^{\sigma}_{a}) = (a) = (0,0), \qquad \text{(III.3.2-25)}$$

$$(C^{\sigma}_{\alpha}) = (\alpha) = (1,0), \qquad \text{(III.3.2-26)}$$

and

$$(C^{0}_{*}) = (C^{1}_{*}) = (e) = (0,1) \qquad \text{(III.3.2-27)}$$

The basis $\{C^{\sigma}_{A}\ |A \in I',\ \sigma \in \mathbb{Z}_2\}$ will now span the doubly graded $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ vector space

$$V(2M_{(0,0)}/1_{(0,1)}/2N_{(1,0)}/0_{(1,1)}) \qquad \text{(III.3.2-28)}$$

where 2M, 1, 2N, and 0 indicates the number of basis vectors (dimensions) in each of the subspace $V_{(0,0)}$, $V_{(0,1)}$, $V_{(1,0)}$ and $V_{(1,1)}$ respectively. Explicitly

$$V_{(0,0)} = \{C^{\sigma}_{a}\} = \{b^{\dagger},\tilde{b}\} \qquad \text{(III.3.2-29a)}$$

$$V_{(0,1)} = \{e\} \qquad \text{(III.3.2-29b)}$$

$$V_{(1,0)} = \{C^{\sigma}_{\alpha}\} = \{f^{\dagger},\tilde{f}\} \qquad \text{(III.3.2-29c)}$$

$$V_{(1,1)} = \phi \quad \text{(empty set)} \qquad \text{(III.3.2-29d)}$$

A supercommutator on the $\mathbb{Z}_2\oplus\mathbb{Z}_2$ graded vector space $V(2M_{(0,0)}/1_{(0,1)}/2N_{(1,0)}/0_{(1,1)})$ may be defined as follows

$$\langle C^{\sigma}_{A}, C^{\tau}_{B}\rangle = C^{\sigma}_{A}\,C^{\tau}_{B} - (-)^{(A)\cdot(B)}\,C^{\tau}_{B}\,C^{\sigma}_{A} \qquad \text{(III.3.2-30)}$$

and the $\mathbb{Z}_2\oplus\mathbb{Z}_2$ graded metric tensor $G^{\sigma\tau}_{AB}$ on the vector space (III.3.2-29) is explicitly given in table III.3.3.

Table III.3.3 : $\mathbb{Z}_2\oplus\mathbb{Z}_2$ graded metric tensor $G^{\sigma\tau}_{AB}$

$G^{\sigma\tau}_{AB}$	C^{0}_{b}	C^{1}_{b}	C^{0}_{β}	C^{1}_{β}	$e/\sqrt{2}$
C^{0}_{a}	0	g_{ab}	0	0	0
C^{1}_{a}	$-g_{ab}$	0	0	0	0
C^{0}_{α}	0	0	0	$g_{\alpha\beta}$	0
C^{1}_{α}	0	0	$-g_{\alpha\beta}$	0	0
$e/\sqrt{2}$	0	0	0	0	1

The direct product space of a $\mathbb{Z}_2\oplus\mathbb{Z}_2$ graded vector space with itself is again a $\mathbb{Z}_2\oplus\mathbb{Z}_2$ graded vector space where a typical element in the space, $C^{\sigma}_{A}\,C^{\sigma}_{B}$ say, has the induced grading

$$(C^{\sigma}_{A}C^{\tau}_{B}) = (C^{\sigma}_{A}) + (C^{\tau}_{B})$$

or $$(A+B) = (A) + (B) \quad \epsilon\ \mathbb{Z}_2 \oplus \mathbb{Z}_2 \tag{III.3.2-31}$$

In particular, consider the operators $S^{\sigma\tau}_{AB}$ which belong to the direct product space, and is defined as a superanticommutator among normalized basis vectors $\{C^{\sigma}_{A}/\sqrt{2} \mid A \in I',\ \sigma \in \mathbb{Z}_2\}$, i.e.

$$\begin{aligned} S^{\sigma\tau}_{AB} &= \langle C^{\sigma}_{A}/\sqrt{2},\ \ C^{\tau}_{B}/\sqrt{2}\rangle_{+} \\ &= (C^{\sigma}_{A}\, C^{\tau}_{B} + (-)^{(A)\cdot(B)} C^{\tau}_{B}\, C^{\sigma}_{A})/2 \end{aligned} \tag{III.3.2-32}$$

$$S^{\sigma\tau}_{AB} = (-)^{(A)\cdot(B)}\ S^{\tau\sigma}_{BA} \qquad \text{(graded symmetric)} \tag{III.3.2-33}$$

where $A,B \in I'$, σ, $\tau \in \mathbb{Z}_2$. The natural grading for $S^{\sigma\tau}_{AB}$ is given as

$$(S^{\sigma\tau}_{AB}) = (A+B) = (A)+(B) \quad \epsilon\ \mathbb{Z}_2 \oplus \mathbb{Z}_2 \tag{III.3.2-34}$$

Explicit content of $S^{\epsilon\tau}_{AB}$ is given in table III.3.5. The evaluation of the supercommutator

$$\langle S^{\epsilon\tau}_{AB},\ S^{\sigma\gamma}_{CD}\rangle = S^{\epsilon\tau}_{AB}\, S^{\sigma\gamma}_{CD} - (-)^{(A+B)\cdot(C+D)}\ S^{\sigma\gamma}_{CD}\, S^{\epsilon\tau}_{AB} \tag{III.3.2-35}$$

proceeds by using the super identities for arbitrary $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ graded operators which was given in (III.2.3-3), and the $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ graded metric tensor $G^{\sigma\tau}_{AB}$ leading to

$$\begin{aligned} \langle S^{\epsilon\tau}_{AB},\ S^{\sigma\gamma}_{CD}\rangle = {} & (-)^{(A)\cdot(B)}\ G^{\epsilon\sigma}_{AC}\, S^{\tau\gamma}_{BD} + (-)^{(A)\cdot(B)+(C)\cdot(D)}\ G^{\epsilon\gamma}_{AD}\, S^{\tau\sigma}_{BC} \\ & + G^{\tau\sigma}_{BC}\, S^{\epsilon\gamma}_{AD} + (-)^{(C)\cdot(D)}\ G^{\tau\gamma}_{BD}\, S^{\epsilon\sigma}_{AC} \end{aligned} \tag{III.3.2-36}$$

In particular, if $D = *$

$$\begin{aligned} \langle S^{\epsilon\tau}_{AB},\ S^{\sigma}_{C*}\rangle &= \langle S^{\epsilon\tau}_{AB},\ C^{\sigma}_{C}/\sqrt{2}\rangle \\ &= (-)^{(A)\cdot(B)}\ G^{\epsilon\sigma}_{AC}\, C^{\tau}_{B}/\sqrt{2} + G^{\tau\sigma}_{BC}\, C^{\epsilon}_{A}/\sqrt{2} \end{aligned} \tag{III.3.2-37}$$

If B = D = $*$

$$\langle S^{\epsilon}_{A*},\ S^{\sigma}_{C*}\rangle = \langle C^{\epsilon}_{A}/\sqrt{2},\ C^{\sigma}_{C}/\sqrt{2}\rangle = S^{\epsilon\sigma}_{AC} \qquad \text{(III.3.2-38)}$$

Thus $S^{\epsilon\tau}_{AB}$ satisfies closure under the supercommutation relation of C(2;s) given in table III.3.1. The symmetry requirement and super Jacobi identity is satisfied automatically as can be seen from the definition of supercommutator and the properties of the superoperators. Hence $\{S^{\sigma\tau}_{AB}$, $A, B \in I'$, $\sigma, \tau \in \mathbb{Z}_2\}$ form the $\mathbb{Z}_2\oplus\mathbb{Z}_2$colour superalgebra C(2;s) which is denoted here as

$$SpO(2M_{(0,0)}/1_{(0,1)}/2N_{(1,0)}/0_{(1,1)})$$

Explicit generators in each sector of the colour superalgebra are given in Table III.3.4. The closure under supercommutator is given symbolically in table III.3.5. The (0,0) and (1,0) sectors $\{S^{\sigma\tau}_{AB}$, $A, B \in I$, $\sigma, \tau \in \mathbb{Z}_2\}$ form the maximum superalgebra, the non-compact SpO(2M/2N), which was given earlier in table III.2.3.

Table III.3.4 : Generators of SpO (2M/1/2N/0) : the linear and bilinear $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ graded operators $S^{\epsilon\tau}_{AB}$

$S^{\epsilon\tau}_{AB}$ \ (B,τ) / (A,ϵ)		$(b,0)$	$(b,1)$	$(\beta,0)$	$(\beta,1)$	$(*,0)$ or $(*,1)$
(A,ϵ)		$C^0_b/\sqrt{2}$	$C^1_b/\sqrt{2}$	$C^0_\beta/\sqrt{2}$	$C^1_\beta/\sqrt{2}$	$e/\sqrt{2}$
$(a,0)$	$C^0_a/\sqrt{2}$	$C^0_a\ C^0_b$	$C^0_a\ C^1_b - (-)^{\ell_a \mp m_a}\delta_{-ab}/2$	$C^0_a\ C^0_\beta$	$C^0_a\ C^1_\beta$	$C^0_a/\sqrt{2}$
$(a,1)$	$C^1_a/\sqrt{2}$	$C^1_a\ C^0_b + (-)^{\ell_b \mp m_b}\delta_{-ba}/2$	$C^1_a\ C^1_b$	$C^1_a\ C^0_\beta$	$C^1_a\ C^1_\beta$	$C^1_a/\sqrt{2}$
$(\alpha,0)$	$C^0_\alpha/\sqrt{2}$	$C^0_\alpha\ C^0_b$	$C^0_\alpha\ C^1_b$	$C^0_\alpha\ C^0_\beta$	$C^0_\alpha\ C^1_\beta - (-)^{j_\alpha \mp m_\alpha}\delta_{-\alpha\beta}/2$	$C^0_\alpha/\sqrt{2}$
$(\alpha,1)$	$C^1_\alpha/\sqrt{2}$	$C^1_\alpha\ C^0_b$	$C^1_\alpha\ C^1_b$	$C^1_\alpha\ C^0_\beta - (-)^{j_\beta \mp m_\beta}\delta_{-\beta\alpha}/2$	$C^1_\alpha\ C^1_\beta$	$C^1_\alpha/\sqrt{2}$
$(*,0)$ $(*,1)$	$e/\sqrt{2}$	$C^0_b/\sqrt{2}$	$C^1_b/\sqrt{2}$	$C^0_\beta/\sqrt{2}$	$C^1_\beta/\sqrt{2}$	0

where $S^{\epsilon\tau}_{AB} = \langle C^\epsilon_A/\sqrt{2},\ C^\tau_B/\sqrt{2}\rangle$, $S^{\epsilon\tau}_{ab} = S^{\tau\epsilon}_{ba}$, $S^{\epsilon\tau}_{a\alpha} = S^{\tau\epsilon}_{\alpha a}$, $S^{\epsilon\tau}_{\alpha\beta} = -\,S^{\tau\epsilon}_{\beta\alpha}$, $S^{\epsilon\tau}_{AB} = (-)^{(A)\cdot(B)}S^{\tau\epsilon}_{BA}$

Note: $L = L_{(0,0)} \oplus L_{(0,1)} \oplus L_{(1,0)} \oplus L_{(1,1)}$ $\qquad \{S^{\epsilon\tau}_{AB}\} = \{S^{\epsilon\tau}_{ab},\ S^{\epsilon\tau}_{\alpha\beta}\} \cup \{S^{\epsilon\tau}_{*a}\} \cup \{S^{\epsilon\tau}_{a\alpha}\} \cup \{S^{\epsilon\tau}_{*\alpha}\}$

Table III.3.5 : Closure of $S^{\epsilon\tau}_{AB}$ under $\mathbb{Z}_2\oplus\mathbb{Z}_2$ graded colour superalgbebra $SpO(2M_{(0,0)}/1_{(0,1)}/2N_{(1,0)}/0_{(1,1)})$

$\langle L_\alpha, L_\beta\rangle$ (L_α \ L_β)		$L_{(0,0)} = \{S^{\epsilon\tau}_{ab}, S^{\epsilon\tau}_{\alpha\beta}\}$						$L_{(0,1)}=\{S^{\epsilon\tau}_{a*}\}$		$L_{(1,0)}=\{S^{\epsilon\tau}_{a\alpha}\}$				$L_{(1,1)}=\{S^{\epsilon\tau}_{\alpha*}\}$	
		$b^\dagger b^\dagger$	$b^\dagger\tilde b$	$\tilde b\,\tilde b$	$f^\dagger f^\dagger$	$f^\dagger\tilde f$	$\tilde f\,\tilde f$	$b^\dagger$	$\tilde b$	$b^\dagger f^\dagger$	$b^\dagger\tilde f$	$\tilde b\, f^\dagger$	$\tilde b\,\tilde f$	$f^\dagger$	$\tilde f$
$L_{(0,0)}$	$b^\dagger b^\dagger$	0	$b^\dagger b^\dagger$	$b^\dagger\tilde b$	0	0	0	0	$b^\dagger$	0	0	$b^\dagger f^\dagger$	$b^\dagger\tilde f$	0	0
	$b^\dagger\tilde b$		$b^\dagger\tilde b$	$\tilde b\,\tilde b$	0	0	0	$b^\dagger$	$\tilde b$	$b^\dagger f^\dagger$	$b^\dagger\tilde f$	$\tilde b f^\dagger$	$\tilde b\,\tilde f$	0	0
	$\tilde b\,\tilde b$			0	0	0	0	$\tilde b$	0	$\tilde b f^\dagger$	$\tilde b\,\tilde f$	0	0	0	0
	$f^\dagger f^\dagger$				0	$f^\dagger f^\dagger$	$f^\dagger\tilde f$	0	0	0	$b^\dagger f^\dagger$	0	$\tilde b f^\dagger$	0	$f^\dagger$
	$f^\dagger\tilde f$					$f^\dagger\tilde f$	$\tilde f\,\tilde f$	0	0	$b^\dagger f^\dagger$	$b^\dagger\tilde f$	$\tilde b f^\dagger$	$\tilde b\,\tilde f$	$f^\dagger$	$\tilde f$
	$\tilde f\,\tilde f$						0	0	0	$b^\dagger\tilde f$	0	$\tilde b\,\tilde f$	0	$\tilde f$	0
$L_{(0,1)}$	$b^\dagger$							$b^\dagger b^\dagger$	$b^\dagger b$	0	0	$f^\dagger$	$\tilde f$	$\tilde b^\dagger f^\dagger$	$b^\dagger\tilde f$
	$\tilde b$								$\tilde b\,\tilde b$	$f^\dagger$	$\tilde f$	0	0	$\tilde b f^\dagger$	$\tilde b\,\tilde f$
$L_{(1,0)}$	$b^\dagger f^\dagger$									0	$b^\dagger b^\dagger$	$f^\dagger f^\dagger$	$f^\dagger\tilde f+b^\dagger\tilde b$	0	$b^\dagger$
	$b^\dagger\tilde f$										0	$b^\dagger\tilde b+f^\dagger f$	$\tilde f\,\tilde f$	$b^\dagger$	0
	$\tilde b f^\dagger$											0	$\tilde b\,\tilde b$	0	$\tilde b$
	$\tilde b\,\tilde f$												0	$\tilde b$	0
$L_{(1,1)}$	$f^\dagger$													$f^\dagger f^\dagger$	$f^\dagger\tilde f$
	$\tilde f$														$\tilde f\,\tilde f$

Note: $SpO(2M_{(0,0)}/1_{(0,1)}/2N_{(1,0)}/0_{(1,1)} = \{S^{\epsilon\tau}_{AB},\ A,B \in I',\ \epsilon,\ \tau \in \mathbb{Z}_2\} = \sum_\alpha L_\alpha,\quad \alpha \in \mathbb{Z}_2\oplus\mathbb{Z}_2.$

The closure under $\langle,\rangle$ is shown symbolically here. Phase factors and indices are ignored. $\tilde b$ $(\tilde f)$ is the modified boson (fermion) annihilation operator.

III.4 SUBALGEBRAS OF THE $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ GRADED COLOUR SUPERALGEBRA SpO(2M/1/2N/0)

In this section we assume that the summation convention for repeated upper and lower indices is adopted unless stated otherwise and $a,b \in B$, $\alpha,\beta \in F$, $A,B \in I$, ϵ, τ, σ, $\zeta \in \mathbb{Z}_2$, where B, F and I are as defined in equation (III.3.2-1)-(III.3.2-3).

The $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ graded colour superalgebra obtained in the last section has a rich algebraic structure which is illustrated in figure FI. It contains C.L.S.A. such as U(M/N), OSp(M/N) and in particular, a generalized quasispin Sp(2,R) as subalgebras. The various chains of subalgebras are established by either discarding selected sets of the generators of the big algebra which is generated by $\{S^{\sigma\tau}_{AB}$, $A,B \in I'$, $\sigma,\tau \in \mathbb{Z}_2\}$ or by forming particular linear combinations of $S^{\sigma\tau}_{AB}$, and projecting the $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ grading vectors down to $\mathbb{Z}_2$ for C.L.S.A. and further to $\mathbb{Z}_1$ for Lie algebras. There are four chains of shown in figure FI.

Chain 1: (Generalized quasispin chain or OSp(M/N) chain)

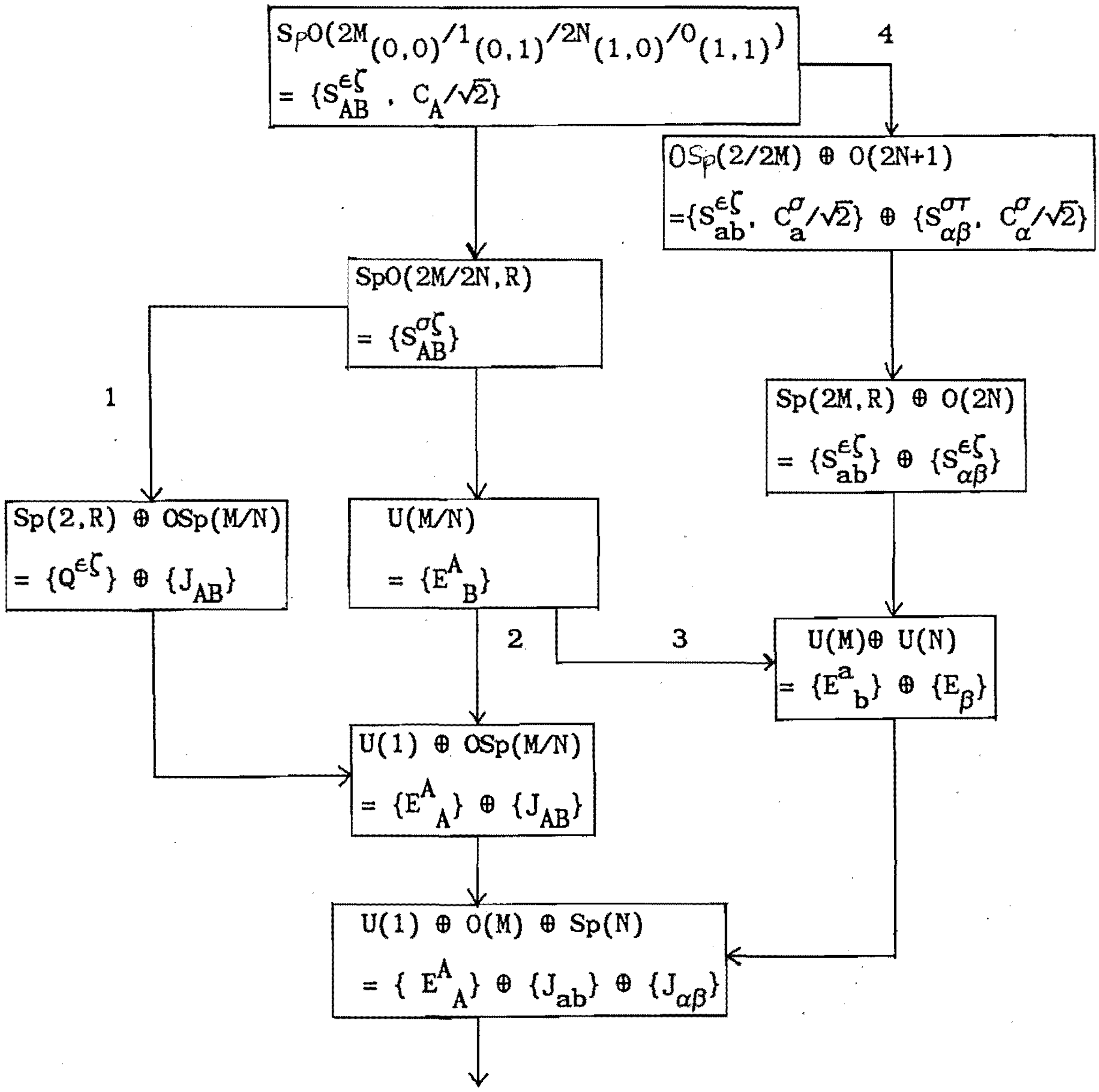
$$\begin{aligned}&SpO(2M_{(0,0)}/1_{(0,1)}/2N_{(1,0)}/0_{(1,1)}) \supset SpO(2M/2N,R)\\ &\supset Sp(2,R)\oplus OSp(M/N) \supset U(1)\oplus OSp(M/N)\\ &\supset U(1)\oplus O(M)\oplus Sp(N) \supset \cdots \qquad \text{(III.4-1)}\end{aligned}$$

Chain 2: (U(M/N) chain)

$$\begin{aligned}&SpO(2M_{(0,0)}/1_{(0,1)}/2N_{(1,0)}/0_{(1,1)}) \supset SpO(2M/2N,R)\\ &\supset U(M/N) \supset U(1)\oplus OSp(M/N) \supset U(1)\oplus O(M)\oplus Sp(N) \supset \cdots\end{aligned}$$

(III.4-2)

Figure I: Subalgebra structures of the $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ graded colour superalgebra.



In this figure

$$a,b \in B, \quad \alpha,\beta \in F, \quad A,B \in I, \quad \epsilon, \sigma = 0 \text{ or } 1$$

$$Q^{\epsilon\zeta} \equiv g^{AB} \, S^{\epsilon\zeta}_{AB}, \quad E^{A}{}_{B} \equiv g^{CA} \, S^{10}_{CB}, \quad J_{AB} \equiv \theta_{\epsilon\zeta} \, S^{\epsilon\zeta}_{AB}$$

and the summation convention is used.

Chain 3: (U(M)⊕U(N) chain)

$$\mathrm{SpO}(2M_{(0,0)}/1_{(0,1)}/2N_{(1,0)}/0_{(1,1)}) \supset \mathrm{Sp}(2M/2N,R)$$
$$\supset \mathrm{U}(M/N) \supset \mathrm{U}(M)\oplus\mathrm{U}(N) \supset \mathrm{U}(1)\oplus\mathrm{O}(M)\oplus\mathrm{Sp}(N) \supset \cdots \tag{III.4-3}$$

Chain 4: (SpO(2/2M)⊕O(2N+1) chain)

$$\mathrm{SpO}(2M_{(0,0)}/1_{(0,1)}/2N_{(1,0)}/0_{(1,1)}) \supset \mathrm{SpO}(2/2M)$$
$$\oplus\mathrm{O}(2N+1) \supset \mathrm{Sp}(2M)\oplus\mathrm{O}(2N) \supset \mathrm{U}(M)\oplus\mathrm{U}(N)$$
$$\supset \mathrm{U}(1)\oplus\mathrm{O}(M)\oplus\mathrm{Sp}(N) \supset \cdots \tag{III.4-4}$$

The *generalized quasispin algebra* Sp(2,R) appears in the first chain whose generators $Q^{\sigma\tau}$ are defined as the contraction of special metric tensor g^{AB} with the bilinear operator $S^{\sigma\tau}_{AB}$ as

$$Q^{\sigma\tau} = \frac{1}{2}\, g^{AB}\, S^{\sigma\tau}_{AB} \tag{III.4-5}$$

symmetry property of $Q^{\sigma\tau}$:

$$Q^{\sigma\tau} = Q^{\tau\sigma} \qquad \text{(symmetric)} \tag{III.4-6}$$

Commutation relation:

$$[Q^{\epsilon\zeta}, Q^{\sigma\tau}] = \frac{1}{2}\,(\theta^{\epsilon\sigma}Q^{\zeta\tau}+\theta^{\epsilon\tau}Q^{\zeta\sigma}+\theta^{\zeta\sigma}Q^{\epsilon\tau}+\theta^{\zeta\tau}Q^{\epsilon\tau}) \tag{III.4-7}$$

where $\theta^{\sigma\tau}, g^{AB}$ were defined earlier in (III.3.2-17 and 20). Expansion of $Q^{\sigma\tau}$ in terms of C^{σ}_{A} gives:

$$Q^{01} = \sum_a \left(\frac{1}{2}(-)^{j_a \mp m_a} C^1_{-a} C^0_a + \frac{1}{4}(2j_a+1)\right)$$

$$+ \sum_\alpha \left(\frac{1}{2}(-)^{j_\alpha \mp m_\alpha} C^1_{-\alpha} C^0_\alpha - \frac{1}{4}(2j_\alpha+1)\right) \qquad \text{(III.4-8a)}$$

$$= \frac{1}{2}(n_b+n_f) + \frac{1}{4}(M-N) \qquad \text{(III.4-8b)}$$

where $n_b(n_f)$ is the boson (fermion) number operator,

$$Q^{11} = \frac{1}{2}\sum_a (-)^{j_a \mp m_a} C^1_a C^1_{-a} - \frac{1}{2}\sum_\alpha (-)^{j_\alpha \mp m_\alpha} C'_\alpha C'_{-\alpha} \qquad \text{(III.4-9)}$$

$$Q^{00} = \frac{1}{2}\sum_a (-)^{j_a \mp m_a} C^o_a C^o_{-a} - \frac{1}{2}\sum_\alpha (-)^{j_\alpha \mp m_\alpha} C^o_\alpha C^o_{-\alpha} \qquad \text{(III.4-10)}$$

$$Q^{10} = Q^{01}$$

and

$$[Q^{01}, Q^{00}] = - Q^{00} \qquad \text{(III.4-11a)}$$

$$[Q^{01}, Q^{11}] = Q^{11} \qquad \text{(III.4-11b)}$$

$$[Q^{11}, Q^{00}] = - 2Q^{01} \qquad \text{(III.4-11c)}$$

If we define

$$Q_o = Q^{01},\ Q_+ = Q^{11},\ Q_- = Q^{00} \qquad \text{(III.4-12)}$$

then the commutation relations above become

$$[Q_o, Q_\pm] = \pm\, Q_\pm \qquad \text{(III.4-13a)}$$

$$[Q_+, Q_-] = -2Q_o \qquad \text{(III.4-13b)}$$

which are exactly the commutation relations for the non-compact Sp(2,R) algebra. We denote the it as $Sp_\pm(2,R)$ where $\mp$ merely signifies the two equivalent sign conventions $(-)^{j_A - m_A}$ and $(-)^{j_A + m_A}$ used in defining the modified annihilation operators. A more detailed study of G.Q.S. is given in section III.5.2.

Suppose we now start from the big algebra and look down along each chain of subalgebras in figure FI, we will find $S^{\epsilon\tau}_{AB}$, $E^{A}{}_{B}$ and J_{AB} appearing successively as generators for SpO(2M/2N,R), U(M/N) and OSp(M/N) respectively. We briefly discuss each case as follows.

(i) $\{S^{\sigma\tau}_{AB},\ A,B \in I,\ \sigma,\tau \in \mathbb{Z}_2\}$ generates the non-compact C.L.S.A. SpO(2M/2N,R). $S^{\epsilon\tau}_{AB}$ is $\mathbb{Z}_2$ graded (since the second component of the original grading vector is always zero) and is bilinear in C^{σ}_{A}. The grading of $S^{\sigma\tau}_{AB}$ is

$$(S^{\sigma\tau}_{AB}) = (A+B) = (A) + (B) \qquad \in \mathbb{Z}_2 \qquad \text{(III.4-14)}$$

where (A), (B) are as defined in (III.3.2-11). Supercommutation relation may be proved by the use of superidentity (III.2.2-3) to be

$$\langle S^{\epsilon\zeta}_{AB}, S^{\sigma\tau}_{CD}\rangle = (-)^{(A)(B)}G^{\epsilon\sigma}_{AB}S^{\zeta\tau}_{BD} + (-)^{(A)(B)+(C)(D)}G^{\epsilon\tau}_{AD}S^{\zeta\sigma}_{BC}$$

$$+ G^{\zeta\sigma}_{BC}S^{\epsilon\tau}_{AD} + (-)^{(C)(D)}G^{\zeta\tau}_{BD}S^{\epsilon\sigma}_{AC} \qquad \text{(III.4-15a)}$$

specially

$$\langle S^{10}_{A-A}, S^{\sigma\tau}_{CD}\rangle = (-)^{(A)}G^{1\sigma}_{AC}S^{0\tau}_{CD} + (-)^{(A)}G^{1\tau}_{AD}S^{\sigma0}_{CD}$$

$$+ G^{0\sigma}_{-AC}S^{1\tau}_{CD} + G^{0\tau}_{-AD}S^{\sigma1}_{CD} \qquad \text{(III.4-15b)}$$

where $G^{\sigma\tau}_{AB}$ is the $\mathbb{Z}_2$ graded metric tensor defined in (III.3.2-13). Express $S^{\sigma\tau}_{AB}$ in terms of C^{σ}_{A} and C^{τ}_{B}

$$S^{\sigma\tau}_{AB} = C^{\sigma}_{A}C^{\tau}_{B} - \frac{1}{2}\,G^{\sigma\tau}_{AB} \qquad \text{(III.4-16a)}$$

$$= (-)^{(A)(B)}C^{\tau}_{B}C^{\sigma}_{A} + \frac{1}{2}\,G^{\sigma\tau}_{AB} \qquad \text{(III.4-16b)}$$

The set of bilinear bosonic operators $\{S^{\sigma\tau}_{ab},\ a,b \in B,\ \sigma,\tau \in \mathbb{Z}_2\}$ generate the non-compact Sp(2M,R) since the following commutation relation is satisfied:

$$[S^{10}_{a-a}, S^{\sigma\tau}_{cd}] = -\theta^{\sigma 1} g_{ac} S^{0\tau}_{cd} - \theta^{\tau 1} g_{ad} S^{\sigma 0}_{cd}$$
$$+ \theta^{0\sigma} g_{-ac} S^{1\tau}_{cd} + \theta^{0\tau} g_{-ad} S^{\sigma 1}_{cd} \qquad \text{(III.4.17)}$$

The set of even bilinear fermionic operators $\{S^{\sigma\tau}_{\alpha\beta}, \alpha, \beta \in F, \sigma,\tau \in \mathbb{Z}_2\}$ span the O(2N) and satisfy

$$[S^{10}_{\alpha-\alpha}, S^{\sigma\tau}_{\gamma\delta}] = \theta^{\sigma 1} g_{\alpha\gamma} S^{0\tau}_{\gamma\delta} + \theta^{\tau 1} g_{\alpha\delta} S^{\sigma 0}_{\gamma\delta}$$
$$+ \theta^{0\sigma} g_{-\alpha\gamma} S^{1\tau}_{\gamma\delta} + \theta^{0\tau} g_{-\alpha\delta} S^{\sigma 1}_{\gamma\delta} \qquad \text{(III.4-18)}$$

We note that $S^{\sigma\sigma}_{\alpha\alpha} = 0$.

Since the grading $(S^{\sigma\tau}_{ab}) = (S^{\sigma\tau}_{\alpha\beta}) = 0$, they are both even and can be regarded as ordinary or $\mathbb{Z}_1$ graded operators.

(ii) Let us define $\mathbb{Z}_2$ graded operator $E^A{}_B$ as

$$E^A{}_B = g^{CA} S^{10}_{CB} \qquad \text{(III.4-19a)}$$
$$= g^{-AA} S^{10}_{-AB} \qquad \text{(no summation)} \qquad \text{(III.4-19b)}$$

Express $E^A{}_B$ in terms of C^1_A and C^0_B:

$$E^A{}_B = g^{-AA} C^1_{-A} C^0_B + \frac{1}{2} (-)^{(A)} \delta_{BA}$$

The grading of $E^A{}_B$ is

$$(E^A{}_B) = (A+B) = (A) + (B) \qquad \in \mathbb{Z}_2 \qquad \text{(III.4-20a)}$$

i.e.

$$(E^a{}_b) = (E^\alpha{}_\beta) = 0 \qquad \text{(even)} \qquad \text{(III.4-20b)}$$
$$(E^\alpha{}_a) = (E^a{}_\alpha) = 1 \qquad \text{(odd)} \qquad \text{(III.4.20c)}$$

the supercommutation relation satisfied by $E^A{}_B$ can be proved by the use of (III.4-15) to be

$$\langle E^A{}_B, E^C{}_D\rangle = \delta^C_B E^A{}_D - (-)^{(A+B)(C+D)}\ \delta^A_D\ E^C{}_B \qquad \text{(III.4.21a)}$$

Particularly,

$$\langle E^A{}_A, E^C{}_D\rangle = \delta^C_A E^C{}_D - \delta^A{}_D E^C{}_D \qquad \text{(III.4-21b)}$$

where

$$\langle E^A{}_B, E^C{}_D\rangle = E^A{}_B E^C{}_D - (-)^{(A+B)(C+D)} E^C{}_D E^A{}_B \qquad \text{(III.4-21c)}$$

Thus $\{E^A{}_B,\ A,B \in I\}$ generates U(M/N). The even, pure boson or fermion operators $\{E^a{}_b\ a,b \in B\}$ and $\{E^\alpha{}_\beta\ \alpha,\beta \in F\}$ generate U(M) and U(N) respectively, where $E^a{}_b$, $E^\alpha{}_\beta$ can be regarded as ordinary ($\mathbb{Z}_1$ graded) operators.

(iii) Let us define J_{AB} as

$$J_{AB} = \theta_{\epsilon\tau} S^{\epsilon\tau}_{AB} \qquad \text{(III.4-22a)}$$

$$= S^{10}_{AB} - S^{01}_{AB} \qquad \text{(III.4-22b)}$$

$$= S^{10}_{AB} + (-)^{(A)(B)} S^{10}_{BA} \qquad \text{(III.4-22c)}$$

Explicitly,

$$J_{AB} = C^1_A C^0_B - C^0_A C^1_B + g_{AB} \qquad \text{(III.4-23a)}$$

or

$$= C^1_A C^0_B - (-)^{(A)(B)} C^1_B C^0_A \qquad \text{(III.4-23b)}$$

Symmetry property of J_{AB}:

$$J_{AB} = -\ (-)^{(A)(B)} J_{BA} \quad \text{(graded antisymmetric)} \qquad \text{(III.4-24)}$$

The grading of J_{AB} is:

$$(J_{AB}) = (A+B) = (A) + (B) \qquad \in \mathbb{Z}_2 \qquad \text{(III.4-25)}$$

Supercommutation relation:

$$\langle J_{AB}, J_{CD}\rangle = -\ (-)^{(A)(B)} g_{AC} J_{BD} + (-)^{(A)(B)+(C)(D)} g_{AD} J_{BC} + g_{BC} J_{AD} - (-)^{(C)(D)} g_{BD} J_{AC} \qquad \text{(III.4-26)}$$

$$\langle J_{A-A}, J_{CD}\rangle = -\,(-)^{(A)(A)}g_{AC}J_{CD} - (-)^{(A)(A)}g_{AD}J_{CD}$$
$$+\, g_{-AC}J_{CD} + g_{-AD}J_{CD} \qquad \text{(III.4-27)}$$

where

$$\langle J_{AB}, J_{CD}\rangle = J_{AB}J_{CD} - (-)^{(A+B)(C+D)}J_{CD}J_{AB} \qquad \text{(III.4-28)}$$

Thus $\{J_{AB}\ A,B \in I\}$ generates th compact C.L.S.A. OSp(M/N). The bosonic part of the even ($\mathbb{Z}_2$ graded) operators: $\{J_{ab}\ a,b \in B\}$ generate O(M) and satisfy the commutation relation

$$[J_{a-a}, J_{cd}] = -\, g_{ac}J_{cd} - g_{ad}J_{cd}$$
$$+\, g_{-ac}J_{cd} + g_{-ad}J_{cd} \qquad \text{(III.4-29)}$$
$$J_{cc} = 0$$

while the fermionic part of the even operator ($\mathbb{Z}_1$ graded): $\{J_{\alpha\beta}\ \alpha_\beta \in F\}$ generate Sp(N) and satisfy

$$[J_{\alpha-\alpha}, J_{\gamma\delta}] = g_{\alpha\gamma}J_{\gamma\sigma} + g_{\alpha\delta}J_{\gamma\sigma}$$
$$+\, g_{-\alpha\gamma}J_{\gamma\delta} + g_{-\alpha\delta}J_{\gamma\delta} \qquad \text{(III.4-30)}$$

The supercommutation relations for $E^A{}_B$ and J_{AB} can be easily evaluated by using the supercommutation relation (III.4.15) satisfied by $S^{\sigma\tau}_{AB}$, since both of these are defined in terms of $S^{\sigma\tau}_{AB}$. The generators in different sectors of a direct sum algebra should be both disjoint and commute with each other. We see that, in most cases, these requirements are trivially satisfied by noticing that one algebra may contain entirely boson operators and the other only fermion operators. For instance, for U(M)⊕U(N), we have

$$U(M)\oplus U(N) = \{E^a{}_b\ a,b \in B\} \cup \{E^\alpha{}_\beta\ \alpha,\beta \in F\} \qquad \text{(III.4-31a)}$$

$$[E^a{}_b,\ E^\alpha{}_\beta] = 0 \qquad \text{(III.4-31b)}$$

$$\{E^a{}_b\} \cap \{E^\alpha{}_\beta\} = \phi\ . \qquad \text{(III.4-31c)}$$

It is also an easy task to show that

$$[Q^{\epsilon\tau}, J_{AB}] = 0 \qquad \text{(III.4-32a)}$$

by first showing

$$[Q^{\epsilon\tau}, C^\sigma_A] = \frac{1}{2}(\theta^{\tau\sigma}C^\epsilon_A + \theta^{\sigma\epsilon}C^\tau_A) \qquad \text{(III.4-32b)}$$

and then making use of (III.4-23) and the superidentity (III.3.2-11).

We have obtained four chains of subalgebras of the $\mathbb{Z}_2\oplus\mathbb{Z}_2$ graded colour superalgebra Sp(2M/1/0/2N). Of particular interest to us is the first chain which contains the quasispin algebra Sp(2,R). Further discussion on G.Q.S. is made in the next section.

III.5 GENERALIZED QUASISPIN

Quasispin formalism is of considerable physical and mathematical importance because it leads to much simpler expressions of matrix elements involving less number of particles and gives the explicit particle number dependence of these elements. The well known ordinary fermionic or bosonic quasispin formalism deals with fermion system and bosonic system separately while the generalized quasispin obtained in the last section involves both bosons and fermions, hence can be used for mixed or 'super' system. Many of the properties of the ordinary quasispin, especially bosonic quasispin, can be carried over to generalized quasispin immediately. We will first review the ordinary quasispin formalism in sectin III.5.1 and then discuss the new generalized quasispin in sectiion III.5.2.

III.5.1 Review of Ordinary Quasispin Formalisms

For a *many fermion system* in a spherical field where single particle states can be characterized by angular momentum $j_\alpha m_\alpha$, such as occurred in the atomic and identical nuclear shell model, fermion quasispin formalism (F.Q.S.) has been proved to be very useful (Lawson and Macfarlane 1965, Arima and Ichimura 1966). Any quantity of physical interest is expressible in this formalism, as a polynomial of fermion creation and modified annihilation operators $f^\dagger_\alpha$, $\tilde{f}_\alpha$, which transform like the two components of a quasispin spinor, hence can be decomposed into a sum of F.Q.S. tensors of various rank k denoted by T^k_q. Basis states of irreps of the F.Q.S. group SU(2) are labelled by $|FF_0JM\rangle$, where F, F_0 is the eigenvalue of F.Q.S. Casimir operator F^2 and its projection F_0 respectively. Due to the complimentary relation between SU(2) and Sp(2j+1) it can be equivalently labelled by $|j^n\upsilon JM\rangle$ where n is the number of fermions in configuration j^n, υ is the number of unpaired fermions (*fermionic seniority number*), J is the total angular momentum and M is its projection, i.e.

$$|j^n\upsilon JM\rangle = |FF_0JM\rangle \qquad \text{(III.5.1-1)}$$

where

$$F = \frac{1}{2}\upsilon - \frac{N}{4} \qquad \text{(III.5.1-2a)}$$

$$F_0 = \frac{1}{2}n - \frac{N}{4} \qquad \text{(III.5.1-2b)}$$

$$N = [j] = 2j+1 \qquad \text{(III.5.2-2c)}$$

We can apply the Wigner-Eckart theorem in F.Q.S. space to the matrix element of F.Q.S. tensor T^k_q to obtain

$$\langle j^n \upsilon JM | T^k_q | j^{n'} \upsilon' J'M' \rangle$$
$$= \Sigma(F'F'_o\ kq | FF_o) \langle j^\upsilon \upsilon JM | T^k | j^{\upsilon'} \upsilon' J'M' \rangle \qquad \text{(III.5.1-3)}$$

Since the fermion number n(n') appears only in the fermion quasispin projection quantum number $F_o(F'_o)$, the n-dependence of the matrix element is factorized out into the Clebsch-Gordan coefficient $(F'F_o'kq|FF_o)$, while the reduced matrix element is entirely independent of n and n'.

With some appropriate modification this formalism can be generalized to mixed configuration $j_1^{n_1} j_2^{n_2} \cdots$ without much difficulty. The complimentary group in this case is Sp(N) where $N = \sum_\alpha (2j_\alpha+1)$.

For a *many-boson system* in a spherical field where a single particle state is labelled by $\ell_a m_a$ such as the many surface phonon states of the nuclei collective model, we have a very similar bosonic quasispin formalism (B.Q.S.) in which the boson number dependence of any matrix element can be separated entirely into the Wigner coupling coefficient for B.Q.S. group SU(1,1) alone and the n-boson problem is reduced to a υ-boson problem, where υ is the *boson seniority number*.

One major algebraic difference between F.Q.S. and B.Q.S. is that the former corresponds to the compact group SU(2) while the latter to the non-compact group SU(1,1). Thus they have different Wigner coupling coefficients (Haruo Ui 1968). The dimensions of unitary irreps are finite for F.Q.S but infinite for B.Q.S.

Properties of fermion quasispin (F.Q.S.) are summarized below. The phase convention adopted here for the modified annihilation operators $\tilde{f}_\alpha$ is $(-)^{j_\alpha - m_\alpha}$ rather than $(-)^{j_\alpha + m_\alpha}$. Summation $\sum_\alpha$ is over all $\alpha \in F$. $[j_\alpha]$ denotes $2j_\alpha+1$. Both $f^\dagger_\alpha$ and $\tilde{f}_\alpha$ are spherical tensor operators of rank j_α. The tensor product of any fermion creation and.or modified annihilation operators is defined as

$$[f^{\epsilon}_{\alpha} \times f^{\tau}_{\beta}]^{J}_{M} = \Sigma\ (j_{\alpha}m_{\alpha}j_{\beta}m_{\beta}|JM)f^{\epsilon}_{\alpha}f^{\tau}_{\beta} \qquad \text{(III.5.1-4)}$$

where $f^{1}_{\alpha} = f^{\dagger}_{\alpha}$, $f^{o}_{\alpha} = \tilde{f}_{\alpha}$; $(j_{\alpha}m_{\alpha}j_{\beta}m_{\beta}|JM)$ is the Clebsch-Gordan coefficient which is familiar from the angular momentum theory. Specially we need to use the result

$$(j_{\alpha}m_{\alpha}j_{\beta}m_{\beta}|00) = \delta_{m_{\alpha}-m_{\beta}}\delta_{j_{\alpha}j_{\beta}}[j_{\alpha}]^{-1/2}(-)^{j_{\alpha}-m_{\alpha}} \qquad \text{(III.5.1-5)}$$

to express the coupled tensor operators in terms of uncoupled operators.

Properties of Fermion Quasispin (F.Q.S.)

Coupled generators:

$$F_{o} = -\frac{1}{4}\sum_{\alpha}[j_{\alpha}]^{1/2}\ ([f^{\dagger}_{\alpha}\tilde{f}_{\alpha}]^{o}_{o} + [\tilde{f}_{\alpha}f^{\dagger}_{\alpha}]^{o}_{o}) \qquad \text{(III.5.1-6a)}$$

$$F_{+} = \frac{1}{2}\sum_{\alpha}[j_{\alpha}]^{1/2}\ [f^{\dagger}_{\alpha}f^{\dagger}_{\alpha}]^{o}_{o} \qquad \text{(III.5.1-6b)}$$

$$F_{-} = -\frac{1}{2}\sum_{\alpha}[j_{\alpha}]^{1/2}\ [\tilde{f}_{\alpha}\tilde{f}_{\alpha}]^{o}_{o} \qquad \text{(III.5.1-6c)}$$

Uncoupled generators:

$$F_{o} = \frac{1}{2}\sum_{\alpha} f^{\dagger}_{\alpha}f_{\alpha} - \frac{N}{4} = \frac{n_{f}}{2} - \frac{N}{4} \qquad \text{(III.5.1-7a)}$$

$$F_{+} = \frac{1}{2}\sum_{\alpha} (-)^{j_{\alpha}-m_{\alpha}}\ f^{\dagger}_{\alpha}f^{\dagger}_{-\alpha} \qquad \text{(III.5.1-7b)}$$

$$F_{-} = -\frac{1}{2}\sum_{\alpha}(-1)^{j_{\alpha}-m_{\alpha}}\ f_{\alpha}f_{-\alpha} \qquad \text{(III.5.1-7c)}$$

Significance of the generators:

F_{o} : proportional to *fermion number operator* which counts the number of fermions.

$F_{\pm}$: *zero coupled fermion pair creation or annihilation operator.* F_{+} creates (F_{-} annihilates) pairs of fermions which are coupled to zero total angular momentum.

Commutation relations:

$$[F_0, F_\pm] = \pm F_\pm \qquad \text{(III.5.1-8a)}$$

$$[F_+, F_-] = 2F_0 \qquad \text{(III.5.1-8b)}$$

or

$$[F_1, F_2] = iF_3 \qquad \text{(III.5.1-9a)}$$

$$[F_2, F_3] = iF_1 \qquad \text{(III.5.1-9b)}$$

$$[F_3, F_1] = iF_2 \qquad \text{(III.5.1-9c)}$$

where $F_3 = F_0$, $F_1 = \frac{1}{2}(F_+ + F_-)$, $F_2 = \frac{1}{2i}(F_+ - F_-)$.

Transformation of $f^+_\alpha, \tilde{f}_\alpha$ under F.Q.S.:

$$[F_0, f^\dagger_\alpha] = \frac{1}{2} f^\dagger_\alpha \qquad \text{(III.5.1-10a)}$$

$$[F_0, \tilde{f}_\alpha] = -\frac{1}{2}\tilde{f}_\alpha \qquad \text{(III.5.1-10b)}$$

$$[F_+, \tilde{f}_\alpha] = f^\dagger_\alpha \qquad \text{(III.5.1-10c)}$$

$$[F_-, f^\dagger_\alpha] = -\tilde{f}_\alpha \qquad \text{(III.5.1-10d)}$$

$$[F_+, f^\dagger_\alpha] = 0 = [F_-, \tilde{f}_\alpha] \qquad \text{(III.5.1-10e)}$$

Thus $(f^\dagger_\alpha, \tilde{f}_\alpha)$ forms a F.Q.S. spinor.

Second order Casimir operator:

$$F^2 = F_1^2 + F_2^2 + F_3^2 \qquad \text{(III.5.1-11a)}$$

$$= F_0(F_0 - 1) + F_+F_- \qquad \text{(III.5.1-11b)}$$

$$[F^2, F_0] = 0 = [F^2, F_\pm] \qquad \text{(III.5.1-12)}$$

F.Q.S. group: Compact SU(2)

A similar list of properties of bosonic quasispin (B.Q.S.) analogous to F.Q.S. is given below. Summation $\sum_a$ is over the boson index set B, i.e. $a \in B$. The two phase conventions $(-)^{j_a \mp m_a}$ which are equal in this case give the identical result.

Properties of Boson Quasispin (B.Q.S.)

Coupled generators:

$$B_o = \frac{1}{4}\,\Sigma[j_a]^{1/2}\,([b_a^\dagger \tilde{b}_a]^o_o + [\tilde{b}_a b_a^\dagger]^o_o) \qquad \text{(III.5.1-13a)}$$

$$B_+ = \frac{1}{2}\,\Sigma[j_a]^{1/2}\,[b_a^\dagger b_a^\dagger]^o_o \qquad \text{(III.5.1-13b)}$$

$$B_- = \frac{1}{2}\,\Sigma[j_a]^{1/2}\,[\tilde{b}_a \tilde{b}_a]^o_o \qquad \text{(III.5.1-13c)}$$

Uncoupled generators:

$$B_o = \frac{1}{2}\sum_\alpha b_a^\dagger b_a + \frac{M}{4} = \frac{n_b}{2} + \frac{M}{4} \qquad \text{(III.5.1-14a)}$$

$$B_+ = \frac{1}{2}\sum_\alpha (-)^{j_a - m_a}\, b_a^\dagger b_{-a}^\dagger \qquad \text{(III.5.1-14b)}$$

$$B_- = \frac{1}{2}\sum_\alpha (-)^{j_a - m_a}\, b_a b_{-a} \qquad \text{(III.5.1-14c)}$$

Significance of the generators:

B_o : is proportional to *boson number operator* which counts the number of bosons.

$B_\pm$: *zero coupled boson pair creation or annihilation operator.* B_+ creates, B_- annihilates pairs of bosons which are coupled to zero total angular momentum.

Commutation relations:

$$[B_o, B_\pm] = \pm\, B_\pm, \qquad \text{(III.5.1-15a)}$$

$$[B_+, B_-] = -\,2B_o, \qquad \text{(III.5.1-15b)}$$

or,

$$[B_1, B_2] = -\,iB_3, \qquad \text{(III.5.1-15c)}$$

$$[B_2, B_3] = iB_1,$$

$$(B_3, B_1] = iB_2$$

where $B_3 = B_o$, $B_1 = \frac{1}{2}(B_+ + B_-)$, $B_2 = \frac{1}{2i}(B_+ - B_-)$.

Transformation of $b^\dagger_a, \tilde{b}_a$ under B.Q.S.:

$$[B_a, b^\dagger_a] = \frac{1}{2} b^\dagger_a, \qquad \text{(III.5.1-16a)}$$

$$[B_o, \tilde{b}_a] = -\frac{1}{2}\tilde{b}_a, \qquad \text{(III.5.1-16b)}$$

$$[B_+, \tilde{b}_a] = -b^\dagger_a, \qquad \text{(III.5.1-16c)}$$

$$[B_-, b^\dagger_a] = \tilde{b}_a, \qquad \text{(III.5.1-16d)}$$

$$[B_+, b^\dagger_a] = 0 = [B_-, \tilde{b}]. \qquad \text{(III.5.1-16e)}$$

thus $(b^\dagger_a, \tilde{b}_a)$ forms a B.Q.S. spinor. Since the group under consideration is non-compact, the finite dimensional tensor obtained above is non-unitary and, moreover, non-Hermitian.

Second order Casimir operator:

$$B^2 = -B_1^2 - B_2^2 + B_3^2 \qquad \text{(III.5.1-17a)}$$

$$= B_o(B_o - 1) - B_+B_- \qquad \text{(III.5.1-17b)}$$

$$[B^2, B_o] = 0 = [B^2, B_\pm] \qquad \text{(III.5.1-18)}$$

F.Q.S. group: non-compact SU(1,1).

III.5.2 Properties of the Generalized Quasispin

The generalized quasispin (G.Q.S.) discovered in section III.4 has very similar properties to the ordinary quasispin. In particular it shares many characteristic properties of the boson quasispin (B.Q.S.) since the two algebras are locally isomorphic to one another. The special appeal of G.Q.S. formalism in dealing with the mixed boson fermion system is entirely analogous to the case of ordinary quasispin formalism for pure fermion or boson system, as discussed in the last section. It is relevant to the applicatiion of supersymmetry concepts to the nuclei system.

In this section we study the properties of G.Q.S. algebra, its representations and its possible applications to nuclei. From the explicit form of generators $Q^{\sigma\tau}$ of G.Q.S. algebra, it is clear that the G.Q.S. is related to the well known bosonic and fermionic quasispin in a very simple way:

$$Q_o = B_o + F_o \tag{III.5.2-1a}$$

$$Q_+ = B_+ \mp F_+ \tag{III.5.2-1b}$$

$$Q_- = B_- \pm F_- \tag{III.5.2-1c}$$

where the top signs in $\mp$, $\pm$ for Q_+, Q_- correspond to the phase choice $(-)^{j_A-m_A}$ in C^o_A while the bottom signs to $(-)^{j_A+m_A}$. The operators $\{Q_o, Q_\pm\}$ form a generalization of the usual F.Q.S. and B.Q.S. algebra with the commutation relations given in (III.4.13). Q_o is proportional to the number operators which counts the number of particles (bosons and fermions alike) in a state. Q_+, Q_- are super zero-coupled pair creation annihilation operators which creates or annihilates pairs of particles (bosons or fermions, but not boson-fermion) which are coupled to zero total angular momentum.

The transformation property of super creation-annhiliation operators C_A under G.Q.S. is

$$[Q^{\epsilon\tau}, C^\sigma_A] = \frac{1}{2}(\theta^{\tau\sigma}C^\epsilon_A + \theta^{\sigma\epsilon}C^\tau_A) \tag{III.5.2-2a}$$

i.e.

$$[Q_o, C^1_A] = \frac{1}{2}C^1_A \tag{III.5.2-3a}$$

$$[Q_o, C^o_A] = -\frac{1}{2}C^o_A \tag{III.5.2-3b}$$

$$[Q_+, C^o_A] = -C^1_A \tag{III.5.2-3c}$$

$$[Q_-, C^1_A] = C^o_A \tag{III.5.2-3d}$$

$$[Q_+, C^1_A] = 0 = [Q_-, C^o_A] \tag{III.5.2-3e}$$

i.e. (C^1_A, C^o_A) transform like the components of G.Q.S. tensor of rank 1/2 or G.Q.S. spinor. Note that the spinor irrep of Sp(2,R) is non-unitary since it is of finite dimension.

By analogy to B.Q.S., the multiplets of irreps of G.Q.S. algebra can be labelled by generalized total quasispin Q and its projection Q_o, as $|Q,Q_o\rangle$. Q is related to the eigenvalue of the second order Casimir generator of G.Q.S., defined as

$$Q^2 = Q_o(Q_o-1) - Q_+Q_- \qquad \text{(III.5.2-4a)}$$

where as usual

$$[Q^2,Q_o] = 0 = [Q^2,Q_\pm] \qquad \text{(III.5.2-4b)}$$

which means all states connected via Q_+, Q_- should have the same Casimir invariants. In order to find more specific results, let us consider the action of $Q_\pm$, Q_o and Q^2 on an arbitrary state of n particles (bosons or fermions) say $|n\rangle$. The action of Q_o on $|n\rangle$ is to count the number of particles. Thus

$$Q_o|n\rangle = (\frac{M-N}{4} + \frac{n}{2})|n\rangle. \qquad \text{(III.5.2-5a)}$$

with

$$Q_o = \frac{M-N}{4} + \frac{n}{2} \qquad \text{(III.5.2-5b)}$$

The action of Q_+, Q_-, as can be seen from the definition of the generators, creates or annihilates pairs of bosons or fermions coupled to zero total angular momentum, changing the number of particles in the state by ±2 and hence the eigenvalue of Q_o by ± 1. $Q_\pm|n\rangle \rightarrow |n\pm2\rangle$.

The fermionic part of the action of Q_+ is restricted by the Pauli exclusion principle but not for the bosonic part. Starting with an initial state $|n\rangle$, Q_+ can be applied repeatedly without limit to connect an infinite set of states.

$$Q^+|n\rangle,\ Q^{+2}|n\rangle\cdots Q^{+k}|n\rangle\cdots$$

while the action of Q_- terminates when a state of υ particles is reached, there being no zero coupled pairs left for Q_- to destroy, i.e.

$$Q_-|\upsilon\rangle = 0 \tag{III.5.2-6}$$

In this case, the G.Q.S. projection is

$$Q_0|\upsilon\rangle = (\frac{M-N}{4} + \frac{\upsilon}{2})|\upsilon\rangle \tag{III.5.2-7}$$

$$Q_0(\min) = (\frac{M-N}{4} + \frac{\upsilon}{2}) \tag{III.5.2-8}$$

which is the lowest value of Q_0 possible. Each G.Q.A. irep hence can be uniquely labelled by the lowest state $|\upsilon\rangle$ where υ is the *generalized seniority number*, the total number of unpaired bosons and fermions.

Consider now the action of Q^2 on $|\upsilon\rangle$. Recalling (III.5.2-4b) we have

$$Q^2|\upsilon\rangle = (Q_0(Q_0-1) - Q_+Q_-)|\upsilon\rangle \tag{III.5.2-9a}$$

$$= (\frac{M-N}{2} + \frac{\upsilon}{2})(\frac{M-N}{2} + \frac{\upsilon}{2} - 1)|\upsilon\rangle \tag{III.5.2-9b}$$

Hence the eigenvalue of Q^2 can be taken as $Q(Q-1)$ where

$$Q = \frac{M-N}{4} + \frac{\upsilon}{2} \geq \frac{M-N}{4} \tag{III.5.2-10}$$

The value of Q may be integer, half integer, quarter integer or three-quarter integers. The eigenvalues of the operator Q^2 are symmetric under (Jucys 1969)

$$Q \rightarrow Q' = Q + 1 \tag{III.5.2-11a}$$

with

$$Q(Q-1) \rightarrow Q'(Q'-1) \geq 0 \tag{III.5.2-11b}$$

hence Q can be always chosen to be positive.

The irreps of the G.Q.S. algebra are uniquely labelled by Q with its basic vectors being labelled as

$$|QQ_o\rangle = \left|\frac{M-N}{4} + \frac{\upsilon}{2}, \frac{M-N}{2} + \frac{n}{2}\right\rangle \qquad \text{(III.5.2-12)}$$

or equivalently by

$$|n\ \upsilon\rangle \quad n = \upsilon + 2i, \quad i = 0,1,2\cdots \qquad \text{(III.5.2-13)}$$

Within each irrep, i.e. for a fixed value of Q, Q_o take the values

$$Q_o = Q,\ Q{+}1,\ Q{+}2,\cdots \qquad \text{(III.5.2-14)}$$

corresponding to the basis states

$$|QQ\rangle,\ |Q,Q{+}1\rangle,\ |Q,Q{+}2\rangle,\cdots \qquad \text{(III.5.2-15)}$$

The multiples of G.Q.S. are shown schematically in figure II.

The G.Q.A. is of importance in physical applications in providing a natural extension of the notions of pairing and seniority to mixed boson-fermion systems. Indeed in the Casimir invariant (III.5.1-7) the term

$$V_{BF.} = F_+B_- + B_+F_- = -\frac{1}{4}\sum_{a,\alpha}(-1)^{\ell_a \mp m_a}(-1)^{j_\alpha \mp m_\alpha}(C^1_\alpha C^1_{-\alpha}C^o_a C^o_{-a} + C^1_a C^1_{-a} C^o_\alpha C^o_{-\alpha})$$

(III.5.2-16)

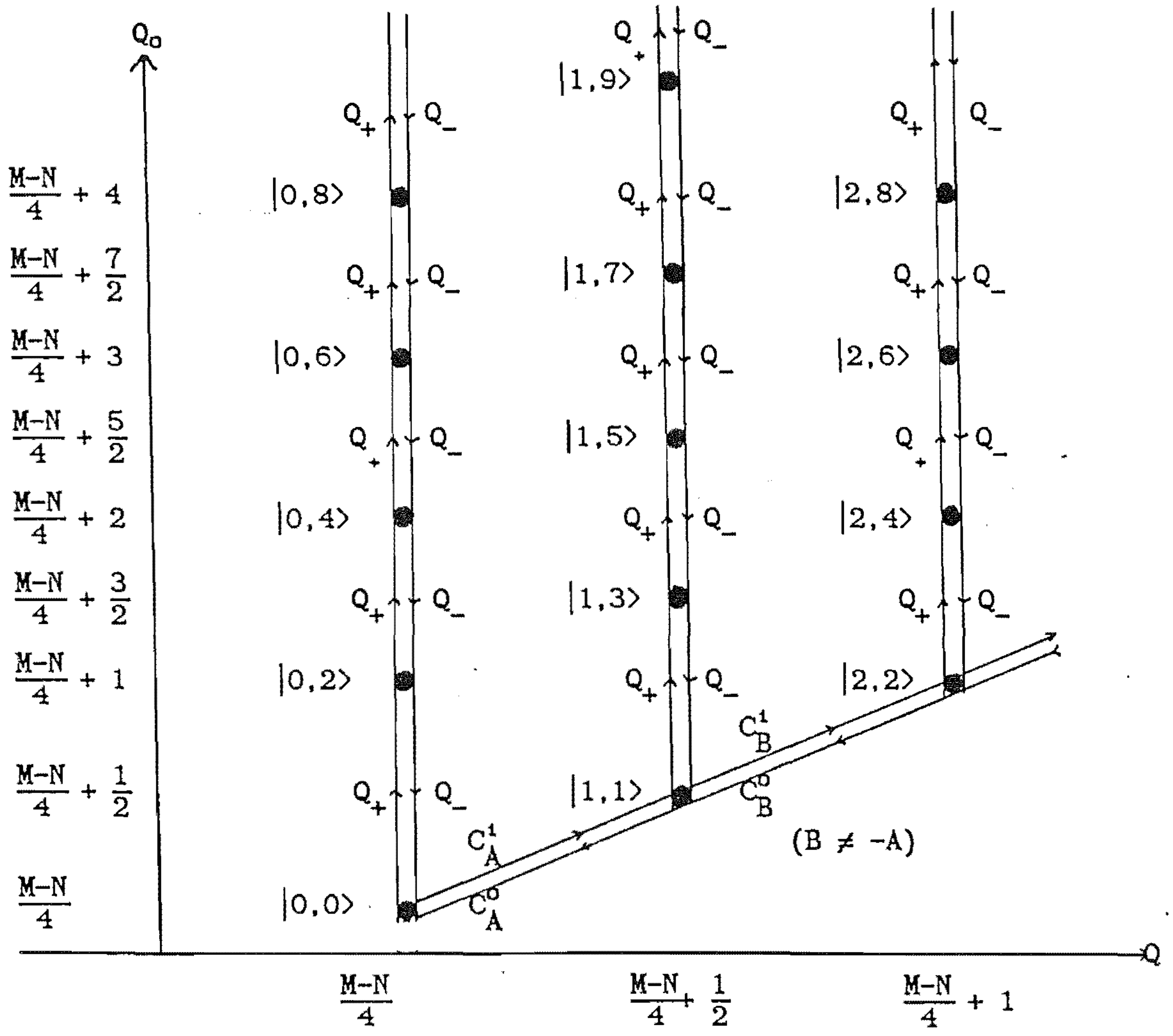
gives via (III.5.2-9) a boson-fermion pairing interaction whose eigenvalues are a function of M, N and υ. Precisely the same V_{BF} was noted in the dynamical supersymmetry scheme based on the U(M/N)⊃Sp(M/N) second-order chain (Morrison and Jarvis 1985). A direct comparison of Q^2 with the second order OSp(M/N) Casimir invariant confirms that

$$Q^2 = \frac{1}{8}J_{AB}J^{BA} + \frac{1}{4}(M-N)(\frac{1}{4}(M-N)-1), \qquad \text{(III.5.2-17)}$$

so that the G.Q.A. gives an alternative and more direct insight into the physics inherent in such models. Beyond this special (exactly soluble) case, the use of G.Q.A. in general permits the explicit n-dependence of matrix elements of interactions to be expressed in term of coupling coefficients via Wigner-Eckart theorem.

Finally it should be pointed out that the two alternative G.Q.A.'s (III.5.2-1) are interchanged by hermitian conjugation. In particular the above V_{BF} is antihermitian, even though Q has real eigenvalues. This situation is perhaps not unexpected if the algebraic models are regarded as reflecting a truncation of the true space of states.

Figure II : Multiplets of the Generalized Quasispin Algebra



In this figure the dots on the same vertical line, starting from some lowest state and unbounded above, belong to a single G.Q.S. multiplet which forms a basis for a unitary irrep of the G.Q.S. algebra. The irrep is characterized by $Q = \frac{M-N}{4} + \frac{\upsilon}{2}$ (x-axis) with $Q(Q-1)$ being the Casimir invariant, and each state within a multiplet specified by $Q_0 = \frac{M-N}{2} + \frac{n}{2} \geq Q$ (y-axis). The arrows and operators besides each line indicate the connections in a multiplet as well as between different multiplets. Each dot (state) is also labelled by $|\upsilon,n\rangle$ where υ is the number of unpaired particles and n the total number of particles in the state.

CONCLUSIONS

We conclude by summarizing the work and results presented in this thesis and discussing avenues for future research.

In chapter I, we have

(a) found more than forty new S-function series and displayed their relations to the L-family series,

(b) developed techniques for identifying contents of S-function series of particular types,

(c) proved general relationships among generating functions for a series (of type I, II or III) and its inverse, conjugate and adjoint series,

(d) proved several identities involving plethysms of S-functions with the L-family series.

In chapter II, we have found

(e) general formulas for calculating plethysms of the basic spin irreps of SO_n for $n \leq 9$,

(f) branching rules for important subgroups of the even dimensional rotation group SO_{2k},

(g) general formulas for the decomposition of the basic spin irrep of SO_{2k} under group-subgroup restriction $SO_{2k} \downarrow SO_{D-2} \times K$ for $D \leq 10$ where K is $SU_N \times U_1$, SO_N, Sp_N, $SO_{N+} \times SO_{N-}$ or $Sp_{N+} \times Sp_{N-}$ for D = 0,4 (mod 8), 1,3(mod 8), 5,7(mod 8), 2(mod 8) or 6(mod 8) respectively,

(h) a method describing the similar decomposition for $D > 10$,

(i) explicit decompositions for $D \leq 11$.

In chapter III, we have

(j) derived four superidentities for evaluating supercommutations of products of graded super operators,

(k) constructed a $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ graded colour superalgebra Sp(2M/1/2N/0) using supercreation-annihilation operators,

(l) identified a non-compact generalized quasispin algebra Sp(2,R),

(m) established various chains of subalgebras of the graded colour superalgebra including the nuclei dynamical supersymmetry algebra U(M/N),

(n) studied the properties of the generalized quasispin.

Although many new series have been found, their significance and applications are yet to be seen. S-function series of other types, e.g. $\prod_{i;j<k}(1-x_i x_j x_k)$ have not been studied apart from knowing that its conjugate series is $\prod_{i;j\leq k}(1+x_i x_j x_k)^{-1}$. Also we are unable to find generating functions for the X, Y and T series. The decomposition of the basic spin irreps under $SO_{2k} \downarrow SO_{D-2} \times K$ has been solved in principle, for all D, but in actual fact, it could only be carried out for $D < 14$. This is because the plethysm of the basic spin irreps of SO_n for $n > 12$ is still unknown. It remains as a very difficult problem to be solved. In chapter III the representations of the colour superalgebra and the various chains of subalgebras have not been examined, nor has the application of the algebraic model to real nuclei be considered, due to the lack of time and available data. They are open for future research.

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